A Nonlinear Dynamics Approach to Evaluating the ‘Realism’ of Food Systems Models

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Abstract

How can theoretical market models—which necessarily abstract from reality—satisfy demands for realism when used to support high-stakes food policy? Past work concludes that modelers can be reasonably required to demonstrate the ‘degree of correspondence’ between a model and reality, but leaves open the question of how to demonstrate correspondence. We suggest that correspondence be demonstrated by requiring modelers to produce persuasive empirical evidence of real-world market dynamics that their models skillfully reproduce. Real-world market dynamics are masked in volatile observed prices. Agricultural economists conventionally attribute price volatility to exogenous random shocks that can be modeled with linear stochastic approaches, but there is increasing recognition that price volatility also may be generated endogenously by nonlinear market dynamics. Selecting between these competing explanations for market instability matters in food policy because they present policymakers very different surrogate realities with divergent policy implications. We propose pre-modeling application of Nonlinear Time Series analysis to distinguish between linear and nonlinear dynamic structure in observed price data, and provide a framework guiding its sound application. Price data testing positive for nonlinear dynamic structure provides evidence that observed market volatility may be explained with parsimonious nonlinear specifications. Alternatively, price data testing negative for nonlinear dynamics provides evidence that linear stochastic approaches may better model observed volatility.

1. Introduction

Models necessarily abstract from a reality to which there is only limited access. At the same time, policymakers rely on these abstractions to regulate real-world food systems. Reliance on unrealistic theoretical models leads to poor policies that "leave the real problem unaddressed, waste resources, and impede learning" [(Saltelli and Funtowitz, 2014), p. 84]. Consequently, Oreskes *et al*. (1994) propose that "where public policy and public safety are at stake, the burden is on the modeler to demonstrate the *degree of correspondence* between the model and the material world it seeks to represent..." (p. 644). The Joint Research Centre of the European Commission formally audits models used in impact assessments of EU initiatives, legislation and policy.

We consider how modelers can satisfy rising demands for realism when their models are used to support high-stakes public food policy. This is a hard question. The philosophy of science literature demonstrates the logical impossibility of verifying models as accurate representations of reality. Models depict open-ended real-world systems in constant flux, whereas the truth of propositions can be established only in closed systems (Oreskes *et al*., 1994). Moreover, the conventional method of verifying the reality of a model by demonstrating a good fit between model output and observed data commits the logical fallacy of 'affirming the consequent': If A, then B; B, therefore A. The fallacy is to ignore that there might be another factor triggering A; in this case, that very different models can be parameterized to provide a good fit (Hornberger and Spear, 1981). At best, a good fit provides circumstantial evidence of realism that must be corroborated with additional evidence excluding other possibilities. Oreskes *et al*. (1994) conclude that modelers can reasonably be required to demonstrate "the *degree of correspondence* between the model and the material world it seeks to represent" when "public policy and public safety are at stake" (p. 644), but leave open the question of how to make this demonstration.

We contend that a reasonable demonstration would provide two lines of evidence: First, modelers conduct pre-modeling data diagnostics to provide empirical evidence of real-world dynamic behavior. Second, modelers demonstrate that diagnosed real-world dynamics are skillfully reproduced by model output. We discuss each of these lines of evidence below.

2. Providing Empirical Evidence of Real-World Food-System Dynamics

Empirical evidence of real-world food-system dynamics is limited to observation of incomplete and noisy time-series data on a limited number of market variables. In practice, modelers must somehow detect real-world dynamics concealed from plain sight in often highly-volatile and random-appearing data.

Agricultural economists conventionally presume that observed market volatility (or 'instability') is due to exogenous random shocks (Belair and Mackey, 1989; Mackey, 1988; Newbery and Stiglitz, 1981). The regular behavior of market data is modeled with linear deterministic equations of motion. Since linear dynamics are limited to exponential or periodic solutions, exogenous random shocks must be added to induce the irregular aperiodic (non-repeated) cyclical behavior characteristic of observed market data (Kantz and Schreiber, 1997).

However, breakthroughs in nonlinear dynamics demonstrate that random shocks are not the only source of dynamic complexity—irregular and complex dynamic behavior can emerge deterministically from simple endogenous nonlinear interactions of system variables (Glendinning, 1994). Agricultural economists increasingly acknowledge the implication that economic volatility can be endogenously generated by nonlinear market dynamics (Berg and Huffaker, 2014; Chavas and Holt, 1993; 1991).

Why should policymakers care whether a model attributes the complexity of observed agricultural systems to exogenous shocks or deterministic nonlinear dynamics? The major reason is that the alternative specifications create very different surrogate realities. A linear-stochastic specification puts policymakers in a surrogate reality where they must guard against random external forces that destabilize the otherwise regular behavior of the regulated system. Effective policies stabilize food prices by reducing exogenous sources of market uncertainty. Discretionary food policies may be counterproductive by interfering with the market's stabilizing forces. Alternatively, a nonlinear-dynamic model puts policymakers in a surrogate reality where they can anticipate and plan for recurrent volatility that emerges systematically. Effective policies smooth food price fluctuations endogenously by reducing market 'frictions' including inelastic demand, inflexible production technologies inhibiting flexible supply responses to market conditions, and lags between investment and adjustment of the production capacities through investments (Berg and Huffaker, 2014). In sum, a model incorrectly attributing observed complexity to random chance stands an increased chance of being discredited by an unanticipated recurring crisis.

How can modelers determine whether random shocks or deterministic nonlinear structure drive real-world market dynamics? We propose application of Nonlinear Time Series analysis (*NLTS*) (Kantz and Schreiber, 1997; Schreiber, 1999)—an emerging empirical arm of nonlinear dynamics—to reconstruct market dynamics from a single observed system variable. Dynamics testing positive for low-dimensional nonlinear dynamic structure provide evidence that observed volatility may be explained with parsimonious nonlinear specifications (Glendinning, 1994). Modelers can experiment with simple nonlinear-feedback structures (Larsen et al., 2014), and reasonably expect to find parsimonious specifications generating agricultural dynamics corresponding to real-world complexity. Alternatively, dynamics testing negative for low-dimensional nonlinear deterministic structure provide evidence that linear stochastic approaches may be a more effective modeling alternative. This paper proposes that *NLTS* be applied before conventional linear autocorrelation techniques that fail to detect nonlinear structure (Sugihara et al., 2012).

How can food-system dynamics be detected in a single observed market variable? The explanation is that any single variable records interactions with other system variables. As Farmer (1987) explained: "…the evolution of [a variable] must be influenced by whatever other variables it's interacting with. Their values must somehow be contained in the history of that thing. Somehow their mark must be there." [(Gleick, 1987), p. 266]. Famous naturalist John Muir intuited this result in the early nineteenth century observing: "When we try to pick something up by itself, we find it hitched to everything else in the universe" (Muir, 1911).

3. A Scheme for Nonlinear Data Diagnostics

A scheme for application of *NLTS* to reconstruct market dynamics from a single observed price series is proposed in Figure 1. Like other time series methods, *NLTS* demands stationary data. Stationarity requires that the "duration of the measurement is long compared to the time scales of the systems" [(Schreiber, 1999), p. 33]. Historic records must be long enough to adequately sample lower-frequency cyclical patterns. *NLTS* also demands informative data. Noisy data obscure real-world behavioral patterns (Kot, 1988). Consequently, the framework initially applies signal processing techniques to prepare data for *NLTS*.

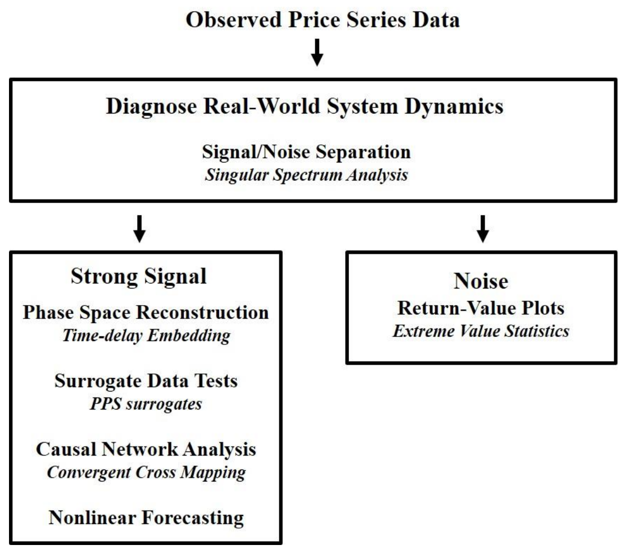

Figure 1. A Framework for Nonlinear Time Series Analysis

3.1 Signal Processing

Signal processing separates data into structured variation (signal) and unstructured variation (noise). Signal strength is measured by the fraction of total variation explained by structured variation. We propose Singular Spectrum Analysis (*SSA*) as a data-adaptive signal-processing method that can accommodate highly anharmonic (potentially non-sinusoidal) oscillations in irregular signals (Elsner and Tsonsis, 2010; Ghil *et al.*, 2002). *SSA* separates an observed price series *p(t)* into the sum of signal (composed of trend and oscillatory components) and residual noise:

$$p(t) = \underbrace{\text{trend} + \text{oscillations}}_{\text{signal}} + \text{noise}\,.$$

3.2 Phase Space Reconstruction

NLTS is applied to reconstruct market dynamics from a sufficiently strong price signal. Consider, for example, the following linear-dynamic market model with two interacting prices $p_x(t)$ and $p_y(t)$:

$$\begin{aligned} p_x(t+1) &= 0.9p_x(t) + 0.3p_y(t) + 100 \qquad (1) \\ p_y(t+1) &= -0.3p_x(t) + 0.9p_y(t) + 100 \end{aligned}$$

The prices solving this dynamic system—plotted as time series in Figure 2A—oscillate toward a steady-state market equilibrium. The market dynamic—pictured in phase space by plotting one price against the other at each point time in Figure 2B—has the two prices co-evolving along the familiar cobweb trajectory toward market equilibrium. The systematic cobweb cycle is an example of a phase-space attractor—a geometric structure with "noticeable regularity" [(Brown, 1996), p. 55]. Depending on system parameters, the linear market system can also produce a stable limit-cycle attractor upon which prices co-

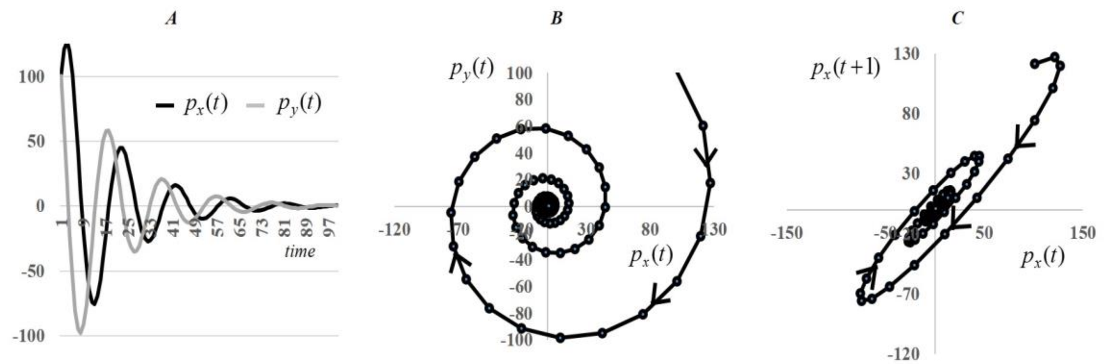

Figure 2. Cobweb price dynamics: (A) Time series plots; (B) Phase-space plot; (C) Reconstructed phase space.

evolve periodically. The market model must be reformulated with nonlinear feedback between the two prices to produce a more complex attractor, for example, one upon which variables cycle aperiodically (i.e., cycles are not repeated) (Glendinning, 1994; Strogatz, 1994).

A 'shadow' version of the cobweb market attractor can be reconstructed by plotting one of the prices, e.g., $p_x(t)$, against its level one period later, $p_x(t+1)$, as in Figure 2C. Two-dimensional phase space in original price coordinates is reproduced by a single price and one of its forward-delayed copies. This is the Time Delay Embedding approach to Phase Space Reconstruction that reconstructs system dynamics (i.e., the cobweb attractor) from a single variable. Takens (1980) derived sufficient conditions guaranteeing time-delay embedding to be a 1:1 mapping of dynamics from original-system to time-delay coordinates (Takens, 1980). This means that the reconstructed shadow market attractor preserves essential dynamic properties of the original attractor.

While the above example demonstrates how system dynamics can be reconstructed from one system variable, it does not demonstrate the full potential of Phase Space Reconstruction since the linear-system cobweb dynamic is already obvious in the time series plots. The utility of Phase Space Reconstruction is to reconstruct nonlinear system dynamics concealed in volatile and random-appearing observed data. Consider, for example, the time series plotted in Figure 3A. It has an irregular appearance that conceals any dynamic structure that might exist. However, reconstructing system dynamics—by plotting the time series against its levels 16 and 32 periods later—uncovers a shadow 'butterfly' attractor (Figure3B) characterizing the long-term dynamics of the three-dimensional nonlinear Lorenz model that was used to generate the time series (Kaplan and Glass, 1995).

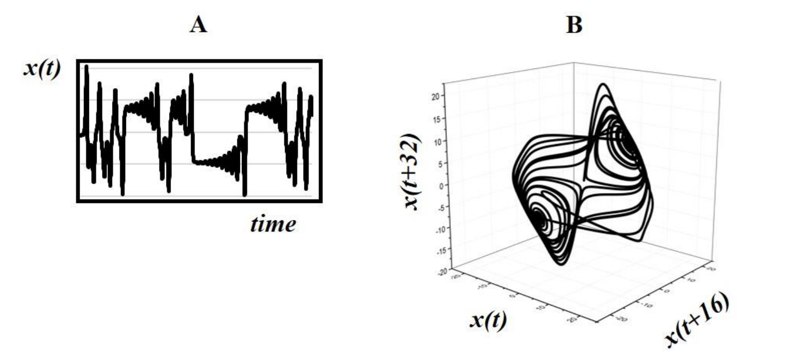

Figure 3. Phase space reconstruction of time series with nonlinear dynamic structure: (A) Time series plot; (B) Reconstructed Lorenz attractor.

In general empirical application, an observed price signal *p(t)* is segmented into a sequence of delay coordinate vectors: *p(t), p(t-d), p(t-2d),.., p(t-(m-1)d)*, where *d* is the 'time delay' and *m* is the number of delayed coordinate vectors (the 'embedding dimension'). The sequence of delay coordinate vectors is collected as columns in an 'embedded data' matrix, and the reconstructed phase space is a scatterplot of the multidimensional points constituting the rows of this matrix:

$$P(t) = \{ p(t), p(t-d), p(t-2d), p(t-(m-1)d \}, \; t = 1,...,T \qquad (2)$$

where T is the terminal time period. The embedding dimension is conventionally selected with the 'false nearest neighbors' test (Williams, 1997). This test measures the percentage of close neighboring points on an attractor in a given embedding dimension that grow apart in the next highest dimension (i.e., 'false neighbors'). The embedding dimension selected for the shadow attractor is that for which the percentage of false neighbors falls below a given tolerance level. The embedding delay is conventionally selected as the first minimum of the 'mutual information function'—a probabilistic measure of the extent to which a variable is related to its delayed value (Williams, 1997). This selection is designed to introduce statistical independence between successive delayed values. Too short a delay does not give system dynamics an adequate opportunity to evolve, while too long a delay causes reconstruction to skip over important dynamic structure.

3.3 Surrogate Data Tests

Surrogate data are used to test the null hypothesis that apparent nonlinear structure visualized in an empirically-reconstructed market attractor is due to a mimicking linear stochastic process (Theiler *et al.*, 1992).

First, a set of surrogate price vectors is randomly generated destroying intertemporal patterns in the observed price signal while preserving various statistical properties. For example, Pseudo Phase Space (*PPS*) surrogates replace the signal's dynamic structure with linearly-filtered noise whose dynamics are characterized by a randomly-shifting limit-cycle attractor (Small and Tse, 2002). Consequently, *PPS* surrogates are used to test the null hypothesis that nonrepeating cycling detected in an empirically-reconstructed attractor is most likely due to a randomly-shifting limit cycle characteristic of stochastic linear dynamics.

Next, an attractor is reconstructed for each surrogate price vector, and a mean 'discriminating measure' taken from the set of surrogate attractors (μ_M) is compared with that taken from the empirically-reconstructed attractor (*M*) by computing a one-sample *t*-statistic:

$$t_{tp} = \frac{\mu_M - M}{SE_M} \qquad (3)$$

where $SE_M = \sigma_M / \sqrt{N}$ is the standard error, and *N* is the length of the price signal. Since a hallmark of deterministic systems is the ability to make short-term predictions of the future, an attractor's predictive skill is a conventional discriminating measure (Small and Tse, 2002; Theiler, *et al.*, 1992). Points on an attractor are divided into forecasting and validation bases. Initially, the nearest neighboring points to the final point in the forecast base, $P(T_F)$, are identified, advanced one time period, and averaged to forecast the first point in the validation base, $P_F(T_F + 1)$. The first coordinate, $p_F(T_F + 1)$, is forecasted price, and the second and third coordinates are lagged forecasted prices. At each step, the forecasting base is augmented by a point in the validation base until all points (excepting the final point) are validated (Kaplan and Glass, 1995; Sprott, 2003). Nash-Sutcliffe Model Efficiency is a conventional discriminating measure of the goodness-of-fit between in-sample predictions and the validation base (Ritter and Munoz-Carpena, 2013):

$$nse = 1 - \frac{\sum_{i=1}^{k} \left(p_k - p_{Fk} \right)^2}{\sum_{i=1}^{k} \left(p_k - \overline{p} \right)^2} \qquad (4)$$

where *k* denotes periods in the validation base, p_k and p_{Fk} are the price signal and its forecasted value in period *k*, respectively, and $\overline{p}$ is the price signal averaged over the validation base. A value $nse = 1$ represents a perfect fit.

An upper-tailed hypothesis test for predictive skill rejects the null hypothesis of linear stochastic dynamics only if the empirically-reconstructed attractor predicts with higher skill than the battery of surrogate attractors on average:

$$\alpha_c \geq 1 - \Phi(t) \qquad (5)$$

where the right-hand side is the *p*-value for an upper-tailed test. Statistically insignificant *p-values* indicate acceptance of the null hypothesis that apparent structure in the empirically-reconstructed attractor is more likely attributed to linearly stochastic behavior. When *PPS* surrogates are used, the particular null hypothesis tested is that nonrepeating cycles detected in an empirically-reconstructed attractor are most likely a single randomly-shifting cycle that can be modeled with a linear stochastic dynamic system, rather than systematic cycling due to low-dimensional nonlinear dynamics.

3.4 Causal Network Analysis

Empirically-reconstructed price attractors can be applied to test whether multiple observed price series' interact in the same causal network, and thus whether they should be codetermined in a partial market model. Recently developed Convergent Cross Mapping (*CCM*) methods test multiple time series' for causal interactions given detection of low-dimensional nonlinear dynamics (Sugihara, *et al.*, 2012). In short, the technique ascertains whether there is pairwise correspondence between reconstructed attractors from two observed time series'. The underlying logic is that causally related variables reconstruct the same real-world attractor. For example, if *Y* drives *X*, then an attractor reconstructed from *X* can be used to estimate ('cross map') values of *Y*, but not vice versa. If *Y* and *X* have a bi-causal relationship, then each can be cross mapped from the reconstructed attractor of the other.

CCM results do not indicate whether interrelated market commodities are uniformly complements or substitutes. A nonlinear data diagnostics approach might appear less powerful and useful than conventional regression and correlation approaches that readily supply this information, but this would be a mistaken perception. When a price series is governed by nonlinear dynamics, linear correlations give ambiguous results because they may shift depending on the time interval during which they are taken (Sugihara, *et al.*, 2012). Consequently, correlations that are used to identify complements and substitutes given nonlinear dynamics are unreliable.

Does this mean that market concepts of complement and substitute goods are obsolete when price series are characterized by nonlinear dynamics? No, but the concepts may need to become more flexible to correspond well to reality. For example, two goods can be complements under some circumstances but not others depending on the availability of other goods. This hypothesis can be tested by building it into the feedback structure of a market model and determining whether a market attractor reconstructed from the simulated data corresponds well to an attractor reconstructed from observed data.

3.5 Nonlinear Forecasting

Section 3.3 discussed how an empirically-reconstructed price attractor can be used to validate nonlinear forecasting in-sample. It also can be used to forecast prices out-of-sample. The forecasting base includes all points on the attractor. At each step, the predicted point is added to the attractor to predict the next point out-of-sample (Kaplan and Glass, 1995).

3.6 Modeling Noise

We propose a novel application of Extreme Value Statistics (Katz, 2010) to model noise separated from an observed price series. Extreme prices occurring in patterns are detected as signal, and those occurring outside of these patterns are detected as unstructured noise. Policymakers confronted with extreme food prices falling outside of structured behavioral patterns benefit from knowing the likelihood of their occurrence.

Noise measures the discrepancy between the observed price series and separated signal. Extreme Value Statistics calculate the likelihood that discrepancies exceed a threshold value within given time intervals. Exceedances follow a Generalized Pareto (*GP*) distribution with two parameters. Of these, the 'shape' parameter, ξ, determines whether the *GP* distribution reduces to a light-tailed exponential distribution $(\xi = 0)$, a heavy-tailed Pareto distribution $(\xi > 0)$, or a bounded beta distribution $(\xi < 0)$. The *GP* distribution provides a good fit to noise when a Q-Q plot (plotting quantiles computed from the fitted distribution against those computed from the noise) falls on a $45^{\bullet}$ line. The quantiles of the fitted *GP* distribution are inverted to solve for a useful noise diagnostic; namely, *return-level plots* estimating the time periods ('return period') expected before a particular unstructured extreme price event ('return level') is realized.

3.7. When Linear Stochastic Modeling is More Effective than NLTS

There are two occasions when conventional linear stochastic modeling is more effective than *NLTS*: 1) When the price signal is weak and observed volatility is attributed best to unstructured noise; and 2) When a strong price signal tests negative for low-dimensional nonlinear dynamics, and consequently, may be inadequately modeled with simple nonlinear feedback structures.

4. Correspondence of Model Output with Reconstructed Food-System Dynamics

An informative test for model specification is whether an attractor constructed from model output is similar to its empirically-reconstructed real-world counterpart. For example, Berg and Huffaker (2014) constructed a nonlinear dynamic market model that faithfully reproduced a torus-type attractor reconstructed from historic German hog-prices. A plausible test for specification of large-scale simulation models is whether dynamics reconstructed from one of the model's variables match a real-world attractor reconstructed from observed data on the variable.

5. Summary and Concluding Comments

We considered how modelers can satisfy rising demands for realism when their models—which necessarily abstract from the real world—are used to support high-stakes public food policy. The philosophy of science literature concludes that modelers can be reasonably required to demonstrate the 'degree of correspondence' between a model and reality, but leaves open the question of how to make this demonstration. We suggested that modelers be required to produce persuasive empirical evidence of real-world market dynamics that their models skillfully approximate.

Empirical evidence of real-world market dynamics is concealed in volatile observed prices. Agricultural economists conventionally attribute price volatility to exogenous random shocks, but others have recognized that price volatility also may be generated endogenously by nonlinear market dynamics. While both linear-stochastic and nonlinear-dynamic models likely can be parameterized to adequately simulate price volatility, the two offer policymakers very different surrogate realities with divergent policy implications. The selection between the two modeling approaches matters in the policy arena.

We proposed pre-modeling application of Nonlinear Time Series analysis (*NLTS*) to test for nonlinear market dynamics in observed price data. We further proposed that *NLTS* be applied before conventional linear autocorrelation techniques since linear diagnostics are not equipped to detect nonlinear dynamic structure, while *NLTS* can detect linear dynamic structure. Detected nonlinear dynamic structure provides evidence that observed market volatility may be explained with parsimonious nonlinear specifications. Alternatively, failure to empirically detect nonlinear dynamics in price data provides evidence that linear stochastic approaches may better explain observed volatility.

We set out a framework for sound pre-modeling application of *NLTS*. The framework boils down to separating signal from noise in observed price data—where signal is composed of systematic dynamic behavioral patterns and noise is composed of unstructured variation—and using a strong signal to empirically reconstruct a phase space attractor that passes surrogate data tests. A statistically significant empirically-reconstructed market attractor provides a geometric picture visualizing systematic real-world market dynamics that a well specified model should be shown to simulate in post-model auditing. The attractor also opens the door to empirical causal network analysis (to identify interacting variables that should be codetermined in a market model), and nonlinear price forecasting. Finally, to complete the diagnostics, we proposed an innovative use of Extreme Value Statistics to stochastically model unstructured noise separated from observed price data. This produces an informative diagnostic estimating the time periods expected before a particular extreme price level occurring outside of regular behavioral patterns is realized.

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