

Describing Income Inequality

Theil Index and Entropy Class Indexes





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by

Lorenzo Giovanni Bellù, Agricultural Policy Support Service, Policy Assistance Division, FAO, Rome, Italy

and

Paolo Liberati, University of Urbino, "Carlo Bo", Institute of Economics, Urbino, Italy

for the

Food and Agriculture Organization of the United Nations, FAO



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1. SUMMARY

This module illustrates the entropy class of inequality indexes. In particular, it shows how different inequality indexes may be obtained by using a general definition (class) of indexes by assigning different values to a fixed parameter. A step-by-step procedure and numerical examples then show how to move from conceptual to operational ground.

2. INTRODUCTION

This tool will deal with a specific class of inequality indexes, the entropy class indexes to which one of the most popular inequality indexes belongs, i.e. the Theil Index.

Objectives

The objective of the tool is to explain the use of complex inequality measures to compare income distributions and to discuss their relative merits as well as their relative disadvantages.

This tool will deal with the entropy class of inequality indexes. It will show how information on inequality may be conveyed by indexes conforming to certain desirable properties¹.

Using complex inequality measures allows us to ranking income distributions according to income inequality. This is particularly useful in an operational context to derive information on the effects of alternative public programs on the distribution of income and on poverty.

Target audience

This module targets current or future policy analysts who want to increase their capacities in measuring impacts of development policies on inequality. On these grounds, economists and practitioners working in public administrations, in NGOs, professional organisations or consulting firms will find this helpful reference material.

Required background

Users should be familiar with basic notions of mathematics and statistics.

Links to relevant EASYPol modules, further readings and references are included both in the footnotes and in [section 8.1](#) of this module².

¹ See EASYPol Module 054: [Policy Impacts on Inequality: Inequality and Axioms for Its Measurement](#).

² EASYPol hyperlinks are shown in blue, as follows:

- training paths are shown in **underlined bold font**;
- other EASYPol modules or complementary EASYPol materials are in ***bold underlined italics***;
- links to the glossary are in **bold**; and

3. CONCEPTUAL BACKGROUND

3.1 General issues

The use of complex inequality measures such as the entropy class can be associated to the use of a descriptive approach to measure inequality. In particular, the use of these indexes does not involve welfare judgments³.

The entropy class of inequality indexes gives a different description of inequality with respect to simple statistical indexes. In particular using complex inequality measures will not give any information about the characteristics of the distribution like location and shape. In fact most of these indexes are translation invariant and they do not say anything about the position of the income distribution.

It is worth noting that much of the discussion will be carried out keeping in mind (either implicitly or explicitly) axioms as an eligible criterion to evaluate the performance of different members of this class of indexes⁴.

3.2 The entropy class of inequality indexes

This class of inequality indexes is based on the concept of «entropy». In thermodynamics, entropy is a measure of disorder. When applied to income distributions, entropy (disorder) has the meaning of deviations from perfect equality.

The definition of a generalised inequality index is the following:

$$[1] \quad E(\alpha) = \frac{1}{n(\alpha^2 - \alpha)} \sum_i \left[\left(\frac{y_i}{\bar{y}} \right)^\alpha - 1 \right]$$

Expression [1] defines a class because the index $E(\alpha)$ assumes different forms depending on the value assigned to α . A positive α captures the sensitivity of the E index to a specific part of the income distribution. With positive and large α , the index E will be more sensitive to what happens in the upper tail of the income distribution. With positive and small α , the index E will be more sensitive to what happens at the bottom tail of the income distribution.

α is a parameter that in principle may range from minus infinity to infinity, i.e. it can take all possible real values. However, from an operational point of view, α is usually

d) external links are in *italics*.

³ See EASYPol Module 050: [*Policy Impacts on Inequality: Welfare Based Measures of Inequality*](#) and EASYPol Modules 001 and 003, respectively: [*Social Welfare Analysis of Income Distributions: Ranking income distributions with Lorenz Curves*](#), and [*Social Welfare Analysis of Income Distributions: Ranking Income Distributions with Generalised Lorenz Curves*](#).

⁴ The reader is therefore strongly advised to look at the EASYPol Module 054: [*Policy Impacts on Inequality: Inequality and Axioms for its Measurement*](#) before proceeding with this module

chosen to be non-negative, as for $\alpha < 0$ this class of indexes is undefined if there are zero incomes.⁵

Two particular cases of [1] are of particular interest for inequality measurement: a) $\alpha=0$; b) $\alpha=1$.⁶

With $\alpha=0$, expression [1] becomes:

$$[2] \quad E(0) = -\frac{1}{n} \sum_i \ln\left(\frac{y_i}{\bar{y}}\right)$$

With $\alpha=1$, expression [1] becomes:

$$[3] \quad E(1) = \frac{1}{n} \sum_i \left(\frac{y_i}{\bar{y}}\right) \ln\left(\frac{y_i}{\bar{y}}\right)$$

$E(0)$ index is called the **mean logarithmic deviation**. $E(1)$ is called the **Theil Index**, by the name of the author who first proposed it in 1967. Both indexes, however, share an undesirable feature, i.e., not being defined if there are zero incomes. Therefore, in a distribution with all zero incomes except for the last, their maximum value cannot be calculated directly. Rather, it can only be calculated by replacing zero incomes with arbitrary «very small incomes». However, if we replaced zero incomes with very small incomes, while $E(1)$ approaches the maximum value of $\ln(n)$ ⁷, the maximum value of $E(0)$ would depend on how small these incomes are defined. In other words, $E(0)$ is not bound).

⁵ Like in formula [1], $\left(\frac{y_i}{\bar{y}}\right)$ raised to a negative power would be an undefined number.

⁶ It is worth noting that expression [1] is not defined for $\alpha=0$ and $\alpha=1$, as the denominator $n(\alpha^2 - \alpha) = 0$ in both cases. Expression [2] and [3] below are therefore calculated by using a rule by *de l'Hôpital*, by which the limit of an undefined ratio between two functions of the same variable is equal to the limit of the ratio of their first derivative. By using the rule by where $\frac{d}{d\alpha} \left(\frac{y_i}{\bar{y}}\right)^\alpha = \left(\frac{y_i}{\bar{y}}\right)^\alpha \ln\left(\frac{y_i}{\bar{y}}\right)$, and

given that $\frac{d}{d\alpha} (n(\alpha^2 - \alpha)) = n(2\alpha - 1)$, this means that we must evaluate: $\lim_{\substack{\alpha \rightarrow 0 \\ \alpha \rightarrow 1}} \frac{\sum_i \left(\frac{y_i}{\bar{y}}\right)^\alpha \ln\left(\frac{y_i}{\bar{y}}\right)}{n(2\alpha - 1)}$, which

gives expressions in the text.

⁷ This can be appreciated through the fact that in the most unequal distribution:

$$E(1) = \frac{1}{n} \left[\approx 0 + \approx 0 + \dots + \left(\frac{y_n}{\bar{y}}\right) \ln\left(\frac{y_n}{\bar{y}}\right) \right] \approx \frac{1}{n} [n \ln(n)] \approx \ln(n).$$

All other members of the class, i.e. for $\alpha > 1$, have as an upper limit $\frac{n^\alpha - n}{n(\alpha^2 - \alpha)}$ ⁸. As can be easily seen, this upper limit depends on α , i.e. it is differentiated among members of the class, and the range of values each member can take does not range between zero and one.

Therefore, for the purpose of an operational approach, it is worth defining a class of **relative entropy inequality indexes** RE , defined as the ratio between the value of the original entropy index and the maximum value each member of that class assumes for any given positive α . This excludes the possibility of considering $E(0)$, as it has not an upper limit. For all other members, i.e. positive α and $\alpha \neq 0$, it is worth defining the following relative indexes:

$$[4] \quad RE(1) = \frac{E(1)}{\max E(1)} = \frac{\frac{1}{n} \sum_i \left(\frac{y_i}{\bar{y}} \right) \ln \left(\frac{y_i}{\bar{y}} \right)}{\ln n}$$

As $E(1)$ has already been defined as the Theil Index, $RE(1)$ can be called the relative Theil Index.

In the same way, we can define a relative generalised entropy index as follows:

$$[5] \quad RE(\alpha) = \frac{E(\alpha)}{\max E(\alpha)} = \frac{\frac{1}{n(\alpha^2 - \alpha)} \sum_i \left[\left(\frac{y_i}{\bar{y}} \right)^\alpha - 1 \right]}{\frac{n^\alpha - n}{n(\alpha^2 - \alpha)}} = \left(\frac{1}{n^\alpha - n} \right) \sum_i \left[\left(\frac{y_i}{\bar{y}} \right)^\alpha - 1 \right]$$

This procedure will ensure that $RE(1)$ and $RE(\alpha)$ will range between zero and one.

4. A STEP-BY-STEP PROCEDURE TO CALCULATE ENTROPY INEQUALITY INDEXES

In order to illustrate the step-by-step procedure required to calculate various entropy inequality indexes, it is worth distinguishing two groups: **a)** $E(0)$, $E(1)$ and $RE(1)$; **b)** $E(\alpha)$ and $RE(\alpha)$ with $\alpha > 1$.

⁸ Since for the most unequal distribution: $E(\alpha) = \left[\frac{1}{n(\alpha^2 - \alpha)} \right] \left[(1-n) + \left(\frac{y_n}{\bar{y}} \right)^\alpha - 1 \right] = \left[\frac{1}{n(\alpha^2 - \alpha)} \right] [n^\alpha - n]$

4.1 A step-by-step procedure for $E(0)$, $E(1)$ and $RE(1)$

Figure 1 illustrates the step-by-step procedure to calculate for the first group of indexes.

Step 1 as usual as us to sort income distributions before proceeding with the calculation of inequality measures.

Step 2 asks us to define the average mean income in the income distribution under analysis.

Step 3 asks us to define, for each income, its ratio with the average level of income in the income distribution as calculated in Step 2.

Step 4 asks us to take the logarithm of each ratio defined in Step 3

Step 5 simply takes the sum of all terms defined in Step 4.

Figure 1: A step-by-step procedure to calculate $E(0)$, $E(1)$ and $RE(1)$

E(0), E(1) and RE(1)	
STEP	Operational content
1	If not already sorted, sort the income distribution by income level
2	Define the average income level in the income distribution
3	Define the ratio between each income and the average income level
4	Define the logarithm of the terms calculated in Step 3
5	Take the sum of the terms calculated in Step 4
6	Divide the sum in Step 5 by n and take its negative, this give $E(0)$
7	Multiply the results in Step 3 by those in Step 4 and take the sum of these values
8	Divide the sum in Step 7 by n , this give $E(1)$
9	Divide $E(1)$ by $\ln n$. This gives $RE(1)$

In **Step 6**, the first index of this family can be calculated by simply dividing the sum taken in Step 3 by the number of observations n and by taking its negative value. This gives $E(0)$.

In **Step 7**, we can proceed with the calculation of another index of this class, by multiplying the results obtained in Step 2 with those obtained in Step 3 and by taking the sum of these products.

In **Step 8**, by dividing the sum obtained in Step 5 by the number of observations n , we get the index $E(1)$.

In **Step 9**, by dividing $E(1)$ by $\ln(n)$ – which is the maximum value of $E(1)$ – we can take the relative Theil Index $RE(1)$.

4.2 A step-by-step procedure for $E(\alpha)$ and $RE(\alpha)$

Figure 2 shows the step-by-step procedure to calculate other members of the entropy class of inequality indexes, when $\alpha > 1$.

The first three steps are identical to those discussed in Figure 1 for the previous case.

Step 4 asks us to choose the value of α .

Step 5 asks us to raise all terms calculated in Step 3 to power α - as chosen in Step 4 – and to subtract 1 from all terms.

Step 6 simply asks us to take the sum of all terms calculated in Step 5.

In **Step 7**, the first inequality index can be calculated, by dividing the sum of Step 6 by the term $n(\alpha^2 - \alpha)$. This gives rise to $E(\alpha)$.

In order to proceed with the calculation of $RE(\alpha)$, instead, we must calculate the maximum value of $E(\alpha)$ in **Step 8**.

By dividing $E(\alpha)$ by the maximum value of $E(\alpha)$, we get $RE(\alpha)$ in **Step 9**.

Figure 2: A step-by-step procedure to calculate $E(\alpha)$ and $RE(\alpha)$.

$E(a)$ and $RE(a)$ with $a > 1$	
STEP	Operational content
1	If not already sorted, sort the income distribution by income level
2	Define the average income level in the income distribution
3	Define the ratio between each income and the average income level
4	Choose a
5	Raise the results in Step 3 to power a and subtract 1
6	Take the sum of the values calculated in Step 5
7	Divide the sum by $n(a^a - 1)$, this gives $E(a)$
8	Calculate the maximum value of $E(a)$
9	Calculate $RE(a)$, by dividing $E(a)$ by $\max E(a)$ as calculated in Step 8

5. AN EXAMPLE OF HOW TO CALCULATE ENTROPY INEQUALITY INDEXES

5.1 A numerical example for $E(0)$, $E(1)$ and $RE(1)$

It is also worth distinguishing, at this stage, the example for $E(0)$, $E(1)$ and $RE(1)$ from the other members of the entropy class.

Table 1: A numerical example to calculate $E(0)$, $E(1)$ and $RE(1)$

STEP 1		STEP 2	STEP 3	STEP 4	STEP 5	STEP 6	STEP 7	STEP 8	STEP 9
Sort the income distribution		Calculate the average level of income	Define the ratio between each income and the average income level	Define the logarithm of Step 3	Take the sum of the terms calculated in Step 4	Divide the sum in Step 5 by n and take its negative, this give $E(0)$	Multiply the results in Step 3 by those in Step 4 and take the sum	Divide the sum in Step 7 by n , this give $E(1)$	Divide the value in Step 8 by the maximum value $\ln(n)$, this gives $RE(1)$
Individual	A - A typical income distribution	Mean income				0.141		0.120	0.074
1	1,000		0.333	-1.099			-0.366		$\ln(5) = 1.6094$ 1.609400
2	2,000		0.667	-0.405			-0.270		
3	3,000		1.000	0.000			0.000		
4	4,000		1.333	0.288			0.384		
5	5,000	3,000	1.667	0.511	-0.706		0.851		
Total individuals (n)	5						0.598		
Total income	15,000								

Table 1 reports the case for $E(0)$, $E(1)$ and $RE(1)$, calculated on a standard income distribution **A**. Steps illustrated with numerical outcomes, closely follow the steps discussed in Figure 1.

Step 2 calculates the average income in the assumed income distribution, which is 3,000 income units. **Step 3** and **Step 4**, instead, provide for the transformations required by [formula \[2\]](#) and [formula \[3\]](#) in the text. The result of this process is a single number (-0.706) in **Step 5**.

The level of $E(0)$ would then be calculated as 0.141 (**Step 6**). Another transformation (**Step 7**) is instead required to calculate $E(1)$. The result of this transformation is again a single number (0.598). This is the basis to calculate $E(1)$ in **Step 8**, which is equal to 0.120.

More important is the calculation of $RE(1)$, i.e. the relative Theil Index in **Step 9**. The outcome (0.074) means that inequality in the simulated income distribution is about 7.4 per cent of the maximum inequality as measured by the Theil Index. Just recall that the maximum level of $E(1)$ is $\ln(n)$. In the specific case, the maximum level of the Theil Index is $\ln(5) = 1.6094$.

5.2 A numerical example for $E(\alpha)$ and $RE(\alpha)$

Table 2 reports a numerical example for the other members of the entropy class, assuming $\alpha=2$.

Table 2: A numerical example to calculate $E(\alpha)$ and $RE(\alpha)$

STEP 1		STEP 2	STEP 3	STEP 4	STEP 5	STEP 6	STEP 7	STEP 8	STEP 9
Sort the income distribution		Calculate the average level of income	Define the ratio between each income and the average income level	Choose $a > 1$	Raise the results of Step 3 to power a and subtract 1	Take the sum of the values calculated in Step 5	Divide the sum in Step 6 by $n(a^2 - a)$. This gives $E(a)$	Calculate the maximum value of $E(a)$	Divide the value in Step 7 by the maximum value in Step 8, this gives $RE(a)$
Individual	A - A typical income distribution	Mean income		a			0.109	2.0	0.055
1	1,000		0.333		-0.889				
2	2,100		0.700		-0.510		$\frac{n(\alpha^2 - \alpha)}{10}$	$\frac{n^\alpha - n}{20}$	
3	2,900		0.967		-0.066				
4	4,000		1.333		0.778				
5	5,000	3,000	1.667	2	1.778	1.091			
Total individuals (n)									
Total income		15,000							

Steps 1 to 3 are exactly the same as those in Table 1. **Step 4** is characteristic of this process, as it asks us to choose α . Once α has been chosen, the numbers calculated in Step 3 must be raised to α . Subtract 1 from the results (**Step 5**). The sum of all these values is a single number (1.111) calculated in **Step 6**. To get $E(\alpha)$, we have to divide the sum of Step 6 for $n(\alpha^2 - \alpha)$. In this specific case, this term is equal to 10. Therefore, the value of $E(\alpha)$ is 0.111 (**Step 7**).

If we want to calculate the relative entropy indexes, the maximum value of the index must be calculated by dividing $n^\alpha - n$ by $n(\alpha^2 - \alpha)$. The first term, in the specific case, is 20; the second, as already noted, is 10. Their ratio is therefore equal to 2 (**Step 8**). **Step 9** reports the calculation of the $RE(\alpha)$ index for $\alpha=2$, which is 0.056. As before, this index means that the measured inequality in the simulated income distribution is about 5.6 per cent of the maximum inequality level.

6. THE MAIN PROPERTIES OF ENTROPY INEQUALITY INDEXES

Some properties are common to all members of the entropy inequality class. Therefore, they will be dealt with jointly.

- **All members of both E and RE class have zero as lower limit.** For $\alpha=0$ and $\alpha=1$, when all incomes are equal, the ratio between each income and mean income is 1. Therefore, $\ln(1)=0$ for all incomes, so that this sum is zero and $E(0)=E(1)=0$. For all other members of the E class, raising the ratio between each income and average income to power α gives a vector of one. Subtracting 1 again gives zero for all incomes. $E(\alpha)$ is therefore zero. This means that the numerators of all RE indexes is also zero.

- **$E(1)$ has $\ln(n)$ as upper limit, while $E(\alpha)$, for $\alpha > 1$ ⁹, has $\frac{n^\alpha - n}{n(\alpha^2 - \alpha)}$ as upper limit.**

As this upper limit depends on α , each member of the E class has its own upper limit. However, *all relative entropy inequality indexes, $RE(\alpha)$, have 1 as upper limit*, as each of them is normalized on the maximum value of $E(\alpha)$.

- **All members of E and RE class are scale invariant.** This is due to the fact that when all incomes are multiplied by a factor β , the ratio between each income and mean income remains the same, as both are multiplied by β .
- **All members of the E and RE class are not translation invariant.** By adding (subtracting) the same amount of money to all incomes, E inequality indexes would decrease (increase). Given that the denominator of RE indexes is constant, they also decrease (increase).
- **All members of the E and RE class satisfy the principle of transfers.** If income is redistributed from relatively richer individuals to relatively poorer individuals, $E(\alpha)$ decreases. The opposite holds true if income is redistributed from relatively poorer to relatively richer individuals. It is worth noting a particular characteristic of the way in which these indexes satisfy the principle of transfers. For $E(0)$, the change in the index depends on both the population size and the level of individual incomes involved in redistribution.¹⁰ In particular, the higher the gap between the income of the receiver and the income of the donor, the greater the reduction of $E(0)$.¹¹ For $E(1)$, the change also depends on the population size and the level of individual incomes involved in redistribution.¹² Finally, for the other members of the entropy class $E(\alpha)$, the change depends not only on the population size and the level of

⁹ Since for $\alpha=0$ and $\alpha=1$ the E index is not defined whenever there is an individual income equal to zero.

¹⁰ This can be seen by considering that: $\frac{dE(0)}{dy_i} = -\left(\frac{1}{n}\right)\left(\frac{1}{y_i}\right)$, which is obtained using the general rule that

$$\frac{d}{dy} \ln f(y) = \frac{f'(y)}{f(y)}.$$

¹¹ Assuming two individuals and the redistribution of a given amount of income dy , the differential of the

index would be: $dE(0) = -\frac{1}{n} \left[\frac{1}{\underbrace{y_1}_{\text{receiver}}} - \frac{1}{\underbrace{y_2}_{\text{donor}}} \right] dy$ under the hypothesis that $dy_2 = -dy_1 \equiv dy$. When y_1 is

very low and y_2 is very high, the difference in square brackets is larger, as the second term in the square brackets is very small. This gives rise to a more negative $dE(0)$.

¹² Again, this can be seen by the derivative: $\frac{dE(1)}{dy_i} = -\frac{1}{n} \left[\frac{1}{y} \left(1 + \ln \left(\frac{y_i}{y} \right) \right) \right]$.

individual incomes but also on level of α .¹³ The relative entropy indexes have a similar behaviour, as the denominator of all these indexes is in fact a constant (the maximum value of the corresponding index).

Table 3 illustrates these properties with examples for $E(0)$, $E(1)$ and $RE(1)$

Table 3: The main properties of $E(0)$, $E(1)$ and $RE(1)$

Individual	A - A typical income distribution	B - Income distribution with equal incomes	C - Income distribution with only one individual having income	Original income distribution with all incomes increased by 20%	Original income distribution with all incomes increased by \$ 2,000	Original income distribution with a redistribution of \$ 100 from the richest to the poorest	Original income distribution with a redistribution of \$ 100 from two individuals around the mean of the income distribution
1	1,000	3,000	0	1,200	3,000	1,100	1,000
2	2,000	3,000	0	2,400	4,000	2,000	2,100
3	3,000	3,000	0	3,600	5,000	3,000	2,900
4	4,000	3,000	0	4,800	6,000	4,000	4,000
5	5,000	3,000	15,000	6,000	7,000	4,900	5,000
Total income	15,000	15,000	15,000	18,000	25,000	15,000	15,000
$E(0)$	0.141	0.000	15.294	0.141	0.043	0.126	0.138
$E(1)$	0.120	0.000	1.609	0.120	0.041	0.109	0.117
$RE(1)$	0.074	0.000	1.000	0.074	0.025	0.068	0.073

UNCHANGED
DECREASED
DECREASED
DECREASED

Obtained by replacing zero incomes with arbitrary small incomes. The value depends on how small the arbitrary incomes are.

Limit value: $\ln(n)$

E indexes decrease less if redistribution occurs between individuals with closer incomes

As we can easily see, the lower limit is zero for all indexes, while the maximum value is 1 only for the relative entropy index $RE(1)$. Note that for $E(0)$ and $E(1)$, the upper limit is calculated by replacing zero incomes with arbitrary small values. But, while $E(0)$ is basically unbounded from above, the maximum level of $E(1)$ tends to 1.609, i.e. $\ln(5)$.

Other properties are illustrated on the right hand side part of Table 3. All indexes have the same value when incomes are increased by, say, 20 per cent; at the same time, all of them decrease when an absolute amount of money is added (2,000 income units in the example).

The last two columns, instead, illustrate how the indexes satisfies the principle of transfers. In general, all of them decrease after a redistribution of money from a relatively richer to a relatively poorer individual. However, this reduction is lower when the income transfer occurs between individuals with closer incomes – compare the last column with the column immediately before it.

Table 4 illustrates the same properties with examples for $E(2)$ and $RE(2)$.

¹³ The derivative, in this case, is: $\frac{dE(\alpha)}{dy_i} = \frac{1}{n(\alpha^2 - \alpha)} \left[\frac{\alpha}{\bar{y}} \left(\frac{y_i}{\bar{y}} \right)^{\alpha-1} \right]$. For the case of $\alpha=2$, the previous

expression collapses to: $\frac{dE(2)}{dy_i} = \frac{1}{n} \left[\frac{1}{\bar{y}} \left(\frac{y_i}{\bar{y}} \right) \right]$.

Table 4: The main properties of $E(2)$ and $RE(2)$

Individual	A - A typical income distribution	B - Income distribution with equal incomes	C - Income distribution with only one individual having income	Original income distribution with all incomes increased by 20%	Original income distribution with all incomes increased by \$ 2,000	Original income distribution with a redistribution of \$ 100 from the richest to the poorest	Original income distribution with a redistribution of \$ 100 from two individuals around the mean of the income distribution
1	1,000	3,000	0	1,200	3,000	1,100	1,000
2	2,000	3,000	0	2,400	4,000	2,000	2,100
3	3,000	3,000	0	3,600	5,000	3,000	2,900
4	4,000	3,000	0	4,800	6,000	4,000	4,000
5	5,000	3,000	15,000	6,000	7,000	4,900	5,000
Total income	15,000	15,000	15,000	18,000	25,000	15,000	15,000
$E(2)$	0.111	0.000	2.000	0.111	0.040	0.102	0.109
$RE(2)$	0.056	0.000	1.000	0.056	0.020	0.051	0.055

↑

This means that inequality in the income distribution is 5.6 per cent of the maximum inequality with $\alpha=2$

↓ UNCHANGED ↓ DECREASED ↓ DECREASED ↓ DECREASED

E indexes decrease less if redistribution occurs between individuals with closer incomes

Things are basically the same as in Table 3. Note again that the indexes decrease less if transfers of income occur among individuals with closer incomes.

Of some importance is also how the RE indexes vary when α varies. This is best observed by recalling the derivative of $E(\alpha)$ with respect to income,

$$\frac{dE(\alpha)}{dy_i} = \frac{1}{n(\alpha^2 - \alpha)} \left[\frac{\alpha}{\bar{y}} \left(\frac{y_i}{\bar{y}} \right)^{\alpha-1} \right]$$

From this expression it is clear that the change of the $RE(\alpha)$ – which is equivalent to the ratio of the change of $E(\alpha)$ on the upper limit of $E(\alpha)$ – depends on the particular situation addressed. In general, this change depends on α , the average income level and also the dispersion of incomes as measured by the ratio between each income level and average income.

7. SYNTHESIS

It is worth having a comparative picture of how the inequality indexes belonging to this class perform with respect to the desirable properties discussed in the previous section. Just recall that these desirable properties are axioms used for inequality measurement¹⁴.

Table 5 reports this comparison, by illustrating how different indexes of the same class behave with respect to desirable properties (axioms).

¹⁴ See EASYPol Module 054: [Policy Impacts on Inequality: Inequality and Axioms for its Measurement](#).

Table 5: Generalised entropy indexes and desirable properties

	LOWER LIMIT	UPPER LIMIT	Principle of transfers	Scale invariance	Translation invariance	Principle of population	Index of relative inequality	Appeal
Entropy $E(0)$	0	The index is not defined when there are zero incomes. It is not bounded from above	YES, more sensible if individuals have distant incomes	YES	NO	YES	YES	Low
Entropy $E(1)$	0	The index is not defined when there are zero incomes. Its limit is $\ln n$	YES, more sensible if individuals have distant incomes	YES	NO	YES	YES	Low
Entropy $E(\alpha)$	0	$\frac{(n^\alpha - n)}{n(\alpha^2 - \alpha)}$	YES, more sensible if individuals have distant incomes	YES	NO	YES	YES	High
Relative entropy $RE(1)$	0	1	YES, more sensible if individuals have distant incomes	YES	NO	NO	NO	Medium
Relative entropy $RE(\alpha)$	0	1	YES, more sensible if individuals have distant incomes	YES	NO	NO	NO	Medium

Note first that the lower limit of all indexes is zero, while the upper limit is very differentiated. Only relative entropy indexes, that are normalised on the maximum value of the corresponding indexes, have 1 as a upper limit.

The behaviour of these indexes is also peculiar with respect to the principle of transfers. All of them respect this principle, but their reaction is differentiated depending on where the income transfer occurs. In particular, all of them react more if the income transfer occurs among two individuals having wider income gaps (not ranks!)¹⁵.

All indexes are scale invariant, and none of them is translation invariant. The implicit concept of inequality is therefore one of relative inequality.

Only entropy indexes, and not relative entropy indexes, satisfy the principle of population, i.e. the invariance of the index to replication of the original population. For this reason, only entropy indexes, and not the relative entropy indexes, belong to the category of Relative Inequality Indexes (RII).

All indexes therefore have shortcomings. The appeal of $E(0)$ and $E(1)$ is low, as they are not defined in the presence of zero incomes. The appeal of $RE(1)$ and $RE(\alpha)$ is medium, as they do not respect the principle of population and do not belong to the class of RII. Therefore, the group of $E(\alpha)$ has the highest appeal, in this class, even though its upper limit depends on the size of the income distribution.

¹⁵ Compare EASYPol Module 040: [Inequality Analysis: The Gini Index](#).

8. READERS' NOTES

8.1 Time requirements

Time requirements to deliver this module is estimated at about three hours.

8.2 EASYPol links

Selected EASYPol modules may be used to strengthen readers' backgrounds and to further expand their knowledge on inequality and inequality measurement.

This module belongs to a set of modules that discuss how to compare, on inequality grounds, alternative income distributions generated by different policy options. It is part of the modules composing a training path addressing [Analysis and monitoring of socio-economic impacts of policies](#).

The following EASYPol modules form a set of materials logically preceding the current module, which can be used to strengthen the user's background:

- ✓ EASYPol Module 000: [Charting Income Inequality: The Lorenz Curve](#)
- ✓ EASYPol Module 001: [Social Welfare Analysis of Income Distributions: Ranking Income Distribution with Lorenz Curves](#)
- ✓ EASYPol Module 040: [Inequality Analysis: The Gini Index](#)
- ✓ EASYPol Module 054: [Policy impacts on inequality: Inequality and Axioms for its Measurement](#)

8.3 Frequently asked questions

- ✓ What is entropy in inequality analysis?
- ✓ What is the advantage of using the entropy class compared with other inequality indexes?

9. REFERENCES AND FURTHER READINGS

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Module metadata

1. EASYPol Module 051

2. Title in original language

English Describing Income Inequality

French

Spanish

Other language

3. Subtitle in original language

English Theil Index and Entropy Class Indexes

French

Spanish

Other language

4. Summary

This module illustrates the entropy class of inequality indexes. In particular, it shows how different inequality indexes may be obtained by using a general definition (class) of indexes by assigning different values to a fixed parameter. A step-by-step procedure and numerical examples then show how to move from conceptual to operational ground.

5. Date

December 2006

6. Author(s)

Lorenzo Giovanni Bellù, Agricultural Policy Support Service, Policy Assistance Division, FAO, Rome, Italy

Paolo Liberati, University of Urbino "Carlo Bo", Institute of Economics, Urbino, Italy

7. Module type

- Thematic overview
- Conceptual and technical materials
- Analytical tools
- Applied materials
- Complementary resources

8. Topic covered by the module

- Agriculture in the macroeconomic context
- Agricultural and sub-sectoral policies
- Agro-industry and food chain policies
- Environment and sustainability
- Institutional and organizational development
- Investment planning and policies
- Poverty and food security
- Regional integration and international trade
- Rural Development

9. Subtopics covered by the module

10. Training path

[Analysis and monitoring of socio-economic impacts of policies](#)

11. Keywords

capacity building, agriculture, agricultural policies, agricultural development, development policies, policy analysis, policy impact analysis, poverty, poor, food security, analytical tool, agricultural policies, income inequality, income distribution, income ranking, welfare measures, entropy class indexes, inequality index, theil index, welfare measures, social welfare functions, social welfare

