EASYPol
Resources for policy making

ANALYTICAL TOOLS

Module 002

Social Welfare Analysis of Income Distributions

Ranking Income Distributions with Generalised Lorenz Curves

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Resources for policy making

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Ranking Income Distributions with Generalised Lorenz Curves

by

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This modules is part of the resource package **Analysis and monitoring of socio-economic impacts of policies.**

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1 SUMMARY

This module illustrates how **Generalised Lorenz (GL) Curves** can be used to identify the best income distribution on social welfare grounds, within a set of alternative income distributions generated by different policy options, in many of the cases where ordinary Lorenz curves fail to work

After illustrating some pitfalls of ordinary **Lorenz Curves**, a cursory presentation of the step-by-step procedure to check for **Generalised Lorenz dominance** and to infer welfare judgements is provided and demonstrated with some simple numerical examples. This module also points out the limitations of the GL approach whenever GL curves cross each other. In addition, it illustrates the need, in some cases, to further restrict the family of admissible **Social Welfare Functions (SWF)** if a unanimous consensus about the ranking of a given set of income distributions has to be obtained. References to applications in a real country case, references to complementary EASYPol modules, notes for trainers and complementary capacity building facilities, are also provided herewith.

2 INTRODUCTION

This module belongs to a set of modules which discuss how to rank different income distributions on welfare grounds that are generated by alternative policy options, such as: private investment support, input subsidies, output protection. this module, is useful in situations where the analyst has to provide information about the likely impact of a policy measure such as a tax/benefit reform, infrastructural investment policy, a specific sectoral or sub-sectoral policy on the distribution of income, more specifically, to answer policy questions such as whether the policy measure under investigation leads to a social welfare improvement or not.

Objectives

The specific objective of this module is to illustrate how GL curves can be used to rank income distributions on welfare grounds, in those cases where ordinary Lorenz curves and related **Atkinson's Theorem** do not enable to infer any welfare judgement about given income distributions.

The user will learn how to make use of GL curves, to draw conclusions on the most preferred income distribution within a set of possible income distributions generated by alternative policy options. He will also learn about the limitations of Generalised Lorenz curves in some cases.

Target audience

This module targets different categories of users in different contexts, for example:

- **trainers** can use this module in capacity development activities e.g. to teach policy analysts how to use household data in policy work.;
- **policy analysts** can use this module as reference material when carrying out their on-the-job tasks;

- **lecturers** in academic courses can use this material to support under-graduate courses in welfare economics, economic policy, development economics and related fields;
- **other users,** such as NGOs, political parties, professional organizations or consulting firms that are willing to enhance their expertise in analyzing welfare impacts of policies by means of analyzing changes in income distributions.

Required background

The trainer is strongly recommended to verify the suitability of the background of the trainees, notably their understanding of the concepts of "income distribution" and "social welfare". In addition, a preliminary knowledge of Lorenz Curves and Lorenz dominance for welfare analysis is required. If this background is weak or missing, the trainer may consider delivering other modules beforehand, as highlighted in the introduction. Other technicalities present in this module can be understood by all people with an elementary knowledge of basic mathematics, statistics and basic principles of calculus. In particular, the user must be familiar with concepts of:

- policy impact simulations
- income distribution
- Lorenz curves and technicalities
- social welfare and social welfare functions

To find relevant materials in these areas, the reader can follow the links included in the text to other EASYPol modules or references [1]. A set of useful links to related EASYPol modules is provided in a section at the end of the document.

3 CONCEPTUAL BACKGROUND: GL CURVES AND SHORROCKS' THEOREM

This section highlights how ordinary Lorenz curves fail to rank income distributions on welfare grounds, whenever ordinary Lorenz curves cross or the Lorenz dominating distribution has a lower mean. Therefore, GL curves are introduced and the Shorrocks' Theorem is presented, which in many cases allows to overcome the limitations of the Lorenz dominance approach.

When attempting to rank income distributions on welfare grounds using Lorenz curves, one the following cases occurs:

- the dominating distribution has a higher or equal mean;
- the dominating distribution has a lower mean;
- there is no domination of one distribution over the other (Lorenz curves cross).

[1] EASYPol hyperlinks are shown in blue, as follows:

a) training paths are shown in **underlined bold font**;

b) other EASYPol modules or complementary EASYPol materials are in ***bold underlined italics***;

c) links to the glossary are in **bold**; and

d) external links are in *italics*.

In the first case, Atkinson's Theorem allows us to conclude that the dominating distribution is welfare-superior; in the other two cases, Atkinson's Theorem does not allow any conclusive judgement about welfare superiority of one distribution with respect to another[2].

Fortunately, in many circumstances, both cases, may be solved by using GL curves, as developed by Shorrocks (1983). The GL curve is obtained as follows:
The *x*-axis records the cumulative proportion of population, as in standard Lorenz curves. Its range is therefore (0,1).

The y-axis records the cumulative mean income, i.e. the mean income is calculated by taking the cumulated income of a given share of the population, divided by the total population[3], as follows:

$$GL\left(\frac{i}{P}\right)=\frac{\sum_{i=1}^{n} y_i}{P}$$

where :
$i=1\ldots n$ is the position of each individual in the income distribution;
P is the total number of individuals in the distribution;
y_i is the income of the i^{th} individual in the distribution
$\sum_{i=1}^{n} y_i$ is the cumulated income up to the i^{th} individual.

GL ordinate range is therefore $(0,\bar{y})$ i.e., the end-point of the GL curve is the mean income of the whole income distribution. This implies that an income distribution with a lower mean income than another distribution, can never be the GL dominating distribution. At least, at the end point, the income distribution with a higher mean income will dominate the one with a lower mean income.

Note the relationship between GL and L curves. GL can also be obtained as the product of the Lorenz curve: $L\left(\frac{i}{P}\right)=\frac{\sum_{i=1}^{n} y_i}{Y}$ times the mean income $\bar{y}=\frac{Y}{P}$, where Y is the total income, as follows:

$$GL\left(\frac{i}{P}\right)=\frac{\sum_{i=1}^{n} y_i}{Y}\times\frac{Y}{P}$$

$$GL\left(\frac{i}{P}\right)=L\left(\frac{i}{P}\right)\times\bar{y}$$

[2] However, Lorenz curves can, of course, be used to measure inequality.

[3] Remember, that ordinary Lorenz curves report the cumulative proportion of income.

There is a useful result linking GL curves and social welfare.

Box 1 - Shorrocks' Theorem (1983)

If the following two conditions are satisfied:

a) the GL curve of distribution **Y** dominates the GL curve of distribution **X**;

b) the decision-maker is income-seeking and inequality-averse (i.e. the SWF has positive first derivative and negative second derivative with respect to individual incomes)

then, social welfare is higher in **Y** than in **X**.

Welfare prescriptions set by Shorrocks are very similar to Atkinson's, but we are now comparing GL curves. This is necessary because Atkinson's results do not cover those cases where the dominating distribution has a lower mean or where Lorenz curves cross[4].

It is worth noting that for all cases where Atkinson's results hold, GL curves and Lorenz curves give the same information, i.e. Lorenz dominance implies and is implied by GL dominance. In addition, for equal mean distributions, whenever Lorenz curves cross, GL curves also cross. This is due to the fact that ordinates of both GLs are obtained by multiplying the ordinates of the Lorenz curves by a constant, i.e., the mean income which is the same for the two distributions.

4 A STEP-BY-STEP PROCEDURE TO RANK DISTRIBUTIONS AND CHECK FOR GL DOMINANCE

Figure 1, below, illustrates the necessary steps to check for GL dominance.

Step 1, as usual, requires that the income distribution be ranked in ascending order.

Step 2 requires that, firstly, the income distributions be checked to see if they have a different mean. If they had the same mean, then GL dominance would be useless.

Steps 3 and **4** require that standard Lorenz curves be built and checked as to whether they cross or whether the dominating distribution has a lower mean income. Only in these two cases, standard Lorenz dominance fails to perform and GL dominance needs to be checked.

[4] Proof of the Shorrocks' Theorem for continuous SWF and continuous income distributions is provided e.g. in Lambert, 1993, pp 62 to 66.

Figure 1 - How to rank income distributions and check for GL dominance

STEP	OPERATIONAL CONTENT
1	Sort income distributions by income level
2	Check whether income distributions have different mean incomes
3	Build Lorenz curves for each distribution
4	Verify that either they cross or that the dominating distribution has a lower mean
5	Build GL curves
6	Check for GL dominance
7	Conclusion: if there is GL dominance, then the dominating distribution has a higher welfare

Once these preliminary steps have been taken, **Step 5** requires that GL curves be built for each income distribution by multiplying standard Lorenz curves by mean incomes. **Step 6** then requires that GL dominance be confirmed. Should there be GL dominance, then the dominating distribution also has a higher welfare (**Step 7**).

5 EXAMPLE OF HOW TO RANK INCOME DISTRIBUTIONS USING GL DOMINANCE

Consider the following simplified example, illustrated in Table 1, below. A given social group composed of five individuals, enjoys income distribution A (Table 1, column c). A specific policy (e.g. improved extension services in agriculture) leads to a change in the income distribution of the five individuals. Thanks to the new policy, individual 2 now enjoys two additional income units, while the other individuals are left unaffected. The resulting income distribution is distribution F (column f).

To check whether this policy leads to a welfare improvement, apply the procedure in the Flowchart 1.

Step 1: The two distributions A and F are already sorted in ascending order (columns c and f).

Step 2: The mean income of the distributions A and F is calculated (columns c and f last row). Note that the mean income of F is greater than that of A (9.0 and 9.4 respectively).

Step 3: The Lorenz curves for the two distributions are calculated. Column c reports the values of the cumulative shares of population (horizontal axis of L curves). Columns d and g report the cumulative shares of income for the A and F distributions respectively (vertical axis of the L curves).

Step 4: Note that the two Lorenz curves cross each other, as reported in Figure 1a, below.

In F, no individual is worse off in absolute terms, compared with A, because all the other incomes are left unchanged. In addition, the mean income has increased, from 9 to 9.4 monetary units. Yet, Atkinson's Theorem does not allow for any welfare comparison between distributions A and F, because there is no Lorenz dominance (Lorenz curves cross each other, as can easily be seen by looking at the cumulative income shares in Table 1, columns d and g). We, therefore, try to rank A and F on welfare grounds looking at GL dominance.

Step 5: The GL curve ordinates are calculated for distributions A and F (columns e and h respectively), and the two GL curves are plotted in Figure 1b.

Step 6: Note that, in Figure 1b, distribution F, GL dominates distribution A, as also is apparent in Table 1,column i, where the difference of the ordinates of the GL curves are reported.

Step 7: Given GL dominance of F over A, and if the decision-maker is income-seeking and inequality-averse, according to the Shorrocks' Theorem, F is welfare-superior to A.

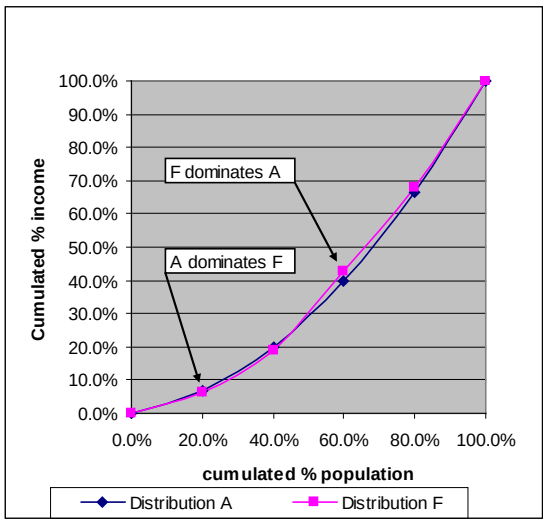
Table 1: Accrual of additional income to an individual and GL dominance

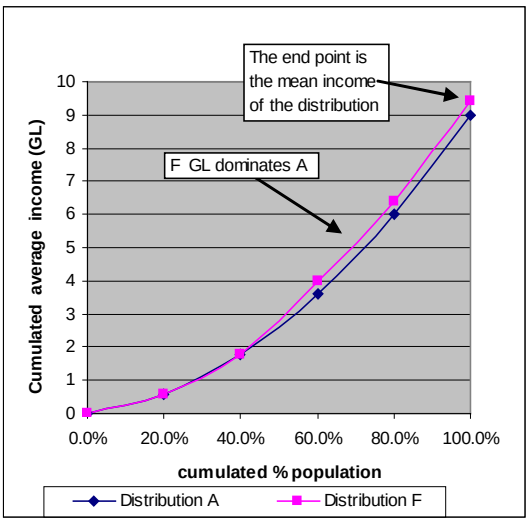
		Distribution A			Distribution F			
Individuals (a)	Cum.share of p (hor.axis L/GL) (b)	Income (Y) (c)	Cum.share Y% (vert.axis of L) (d)	Cum.aver.Y (vert.axis GL) (e)	Income (Y) (f)	Cum.sh.Y% (vert.axis of L) (g)	Cum.aver.Y (vert.axis GL) (h)	Diff.cum. aver.Y F-A (i)
1	20.0%	3	6.7%	0.6	3	6.4%	0.6	0.0
2	40.0%	6	20.0%	1.8	6	19.1%	1.8	0.0
3	60.0%	9	40.0%	3.6	11	42.6%	4.0	0.4
4	80.0%	12	66.7%	6.0	12	68.1%	6.4	0.4
5	100.0%	15	100.0%	9.0	15	100.0%	9.4	0.4
Total income		45			47			
Mean income		9.0			9.4			

Remark: The cumulative average income, say, the ordinate of the GL curve, can also be obtained multyplying the cumulative share of income (the L ordinate) times the mean income of the distrib.

Remark: In F cumulated average incomes below the level of income increased are unchanged w.r.t. A. Those equal or above are greater. Generalised Lorenz curve of F therefore dominates.

Figures 1a and 1b - Additional income accruing to an individual and GL dominance





Welfare superiority of F, as signalled by its GL dominance, is implied by the fact that the SWF of the decision-maker, as made explicit in Shorrocks' Theorem, not only favours transfers from richer to poorer, but also favours income increases. Thus, even in the absence of transfers, additional income accruing to an individual without damaging any other individual is good news for the decision-maker.

In this case, the L curve fails to detect welfare superiority of F because it is based on cumulative shares of incomes. The first individual and the first two aggregated individuals receive a lower cumulative share in F than in A, while the first three aggregated individuals receive a greater cumulative share in F (compare columns d and g in Table 1, above, causing crossing of L curves. Therefore, crossing of Lorenz curves can also occur when there is a Pareto improvement in the income distribution, as in F, and thus fails to perform as indicators of welfare superiority. Whereas, in these cases, GL curves allow us to draw welfare judgements.

5.1 Further examples

In the following example, distribution H is derived from distribution A, reducing by two units the income of the fifth individual. Table 2, below, illustrates the two distributions and related L and GL ordinates. The arrow indicates the income change in distribution H with respect to distribution A. The L and GL curves are illustrated in Figure 2a and b, below, respectively.

Table 2 - GL Inferiority occurring with Lorenz dominance

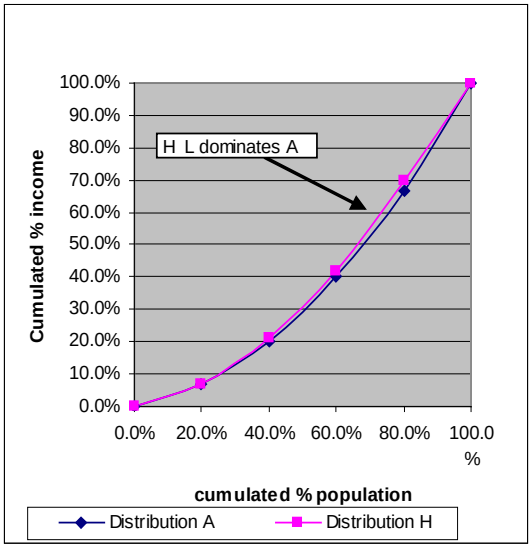
		Distribution A			Distribution H			
Individuals (a)	Cum.share of p (hor.axis L/GL) (b)	Income (Y) (c)	Cum.share Y% (vert.axis of L) (d)	Cum.aver.Y (vert.axis GL) (e)	Income (Y) (f)	Cum.sh.Y% (vert.axis of L) (g)	Cum.aver.Y (vert.axis GL) (h)	Diff.cum. aver.Y H-A (i)
1	20.0%	3	6.7%	0.6	3	7.0%	0.6	0.0
2	40.0%	6	20.0%	1.8	6	20.9%	1.8	0.0
3	60.0%	9	40.0%	3.6	9	41.9%	3.6	0.0
4	80.0%	12	66.7%	6.0	12	69.8%	6.0	0.0
5	100.0%	15	100.0%	9.0	13	100.0%	8.6	-0.4
Total income		45			43			
Mean income		9.0			8.6			

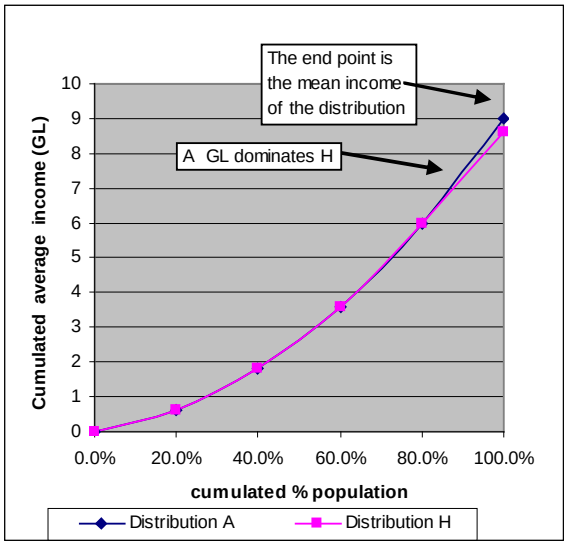
H, L dominates A although incomes in the lower part of the distribution are the same. This is due to the reduction of income in the highest part of the dstribution which leads to an increase of the shares in the lower part of the distribution.

Note that the GL do not cross because the difference between the ordinates of H and A are always either 0 or negative

Note that H L dominates A and incomes are more equally distributed. On the other hand, the mean income drops from 9.0 to 8.6 units. H is GL dominated by A. Therefore, according to Shorrocks' Theorem, H is welfare inferior to A. Note than in H, nobody is better off than in A; in addition, the fifth individual is worse-off. Therefore, even if the income is more equally distributed, as signalled by its L dominance, H represents a "Pareto worsening" with respect to A. Welfare inferiority of H is due to the fact that we are assuming the point of view of a decision-maker who is not only inequality-averse but is also an income-seeker.

Figures 2a and 2b - GL inferiority occurring with Lorenz dominance





In Table 3, below, a further example is considered. Distribution I is the result of a policy the net impact of which results in mixed shifts of income from richer to poorer, i.e. one income unit from individual 3 to individual 1, and from poorer to richer, i.e. one unit of

income from individual 4 to individual 5. Use L curves to check whether A is welfare superior to I.

Table 3 - Mixed transfers from richer to poorer and from poorer to richer

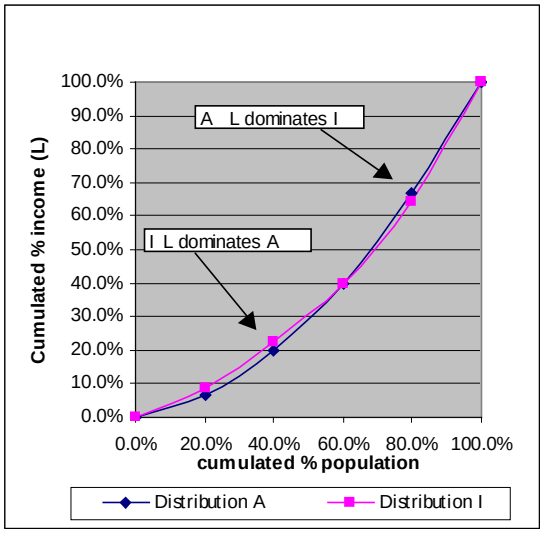
		Distribution A			Distribution I			
Individuals (a)	Cum.share of p (hor.axis L/GL) (b)	Income (Y) (c)	Cum.share Y% (vert.axis of L) (d)	Cum.aver.Y (vert.axis GL) (e)	Income (Y) (f)	Cum.sh.Y% (vert.axis of L) (g)	Cum.aver.Y (vert.axis GL) (h)	Diff.cum. aver.Y H-A (i)
1	20.0%	3	6.7%	0.6	4	8.9%	0.8	0.2
2	40.0%	6	20.0%	1.8	6	22.2%	2.0	0.2
3	60.0%	9	40.0%	3.6	8	40.0%	3.6	0.0
4	80.0%	12	66.7%	6.0	11	64.4%	5.8	-0.2
5	100.0%	15	100.0%	9.0	16	100.0%	9.0	0.0
Total income		45.0			45.0			
Mean income		9.0			9.0			

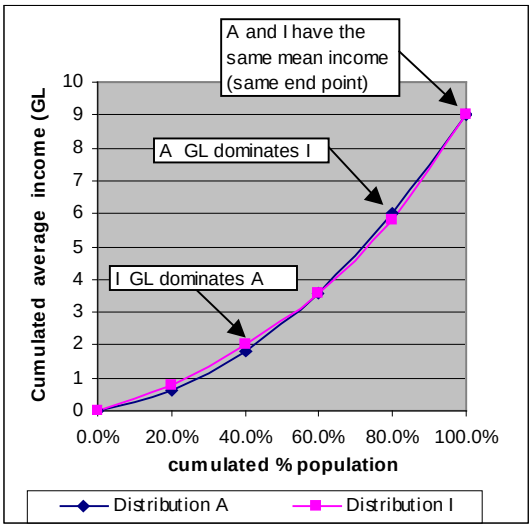
I L dominates A for the first 60% of the population but A L dominates I for greater cumulated shares of the population, i.e. I presents lower cumulated shares of income in the lower part of the distribution and higher cumulated shares in the higher part of the distribution. Therefore, L curves cross.

Note that the GL do cross because the difference between the ordinates of I and A are positive in the lower part of the curves and negative in the upper part.

In this case, L curves cross, as is apparent from Figures 3a and 3b, below, and Table 3, columns d and g,,above. Unfortunately, GL curves also cross. This is not surprising indeed, because distributions A and I have the same mean income. In this case, GL curves are simply "up-scaled" Lorenz curves. Therefore, no conclusive judgement can be reached at this point[5].

Figures 3a and 3b - Mixed transfers from richer to poorer and from poorer to richer





[5] The issue of GL curves crossings is analysed in the EASYPol Module 003: Social Welfare Analysis of Income Distributions: Ranking Income Distributions with Crossing Generalised Lorenz Curves.

6 CONCLUSION

It is worth summarising the main results achieved so far. The basic result is that both Lorenz curves, and GL curves, thanks to both Atkinson's and Shorrocks' Theorems, are powerful tools for ranking different income distributions on welfare grounds. In many cases, when Lorenz curves fail to provide a conclusive answer, GL curves can succeed.

However, unlike the case of the complete specification of a SWF, these tools may give a «partial ordering» of a set of income distributions, as there might be cases where both Lorenz and GL curves do not allow any conclusive welfare judgement, as seen for example in the latter case, where GL curves cross.

Table 4, below, summarises all results achieved so far, highlighting all outcomes deriving from the combination of the type of relationship between curves and mean incomes of the distribution observed.

Table 4 - Summary of results

#	Type of Dominance	Mean income	Welfare ranking	Restrictions on The SWF	Notes
1	L(Y)>L(X)	Y=X	W(Y) > W(X)	Wi' > 0; Wi'' < 0	
2	L(Y)>L(X)	Y>X	W(Y) > W(X)	Wi' > 0; Wi'' < 0	
3	L(Y)>L(X)	Y<X	Cannot say		Need GL
4	L(Y) and L(X) cross	Whatever	Cannot say		Need GL
5	GL(Y) < GL(X)	Y<X	W(Y) < W(X)	Wi' > 0; Wi'' < 0	Can solve # 3
6	GL(Y) > GL(X)	Y=X	W(Y) > W(X)	Wi' > 0; Wi'' < 0	Can solve # 4
7	GL(Y) > GL(X)	Y>X	W(Y) > W(X)	Wi' > 0; Wi'' < 0	Can solve # 4
8	GL(Y) > GL(X)	Y<X	Cannot occur		
9	GL(Y) and GL(X) cross	Whatever	Cannot say		Need further restrictions

Legenda
L(Y): Lorenz Curve of distribution Y
L(X): Lorenz Curve of distribution X
W(X): Social Welfare in X
W(Y): Social Welfare in Y
Wi' and Wi'': First and second derivative respectively of W w.r.t. the income of the i-th individual
GL(Y): Generalised Lorenz Curve of distribution Y
GL(X): Generalised Lorenz Curve of distribution X

It is worth noting again three important aspects:

- GL curves are required when either Lorenz curves cross or the dominating distribution has the lower mean (cases **3** and **4**);
- Case 8 cannot occur, because the end point of GL is the mean income;
- When GL curves cross, additional restrictions on the form of W are required in any case (case **9**).

7 READERS' NOTES

7.1 Time requirements

The delivery of this module and related discussion may take two to three hours to an audience already familiar with concepts of policy, policy impact simulations, income and income distributions, Lorenz curves, social welfare and social welfare functions.

7.2 Frequently asked questions

Frequently asked questions are e.g. the following:

- ✓ **What is the meaning and role of the preferences of the decision-maker?** i.e., what does it mean that the decision-maker is "inequality-averse" and an income-seeker? It is important in these cases to refer to the shape of the welfare function imposed by the restrictions on its first and second derivatives.
- ✓ Why do Lorenz curves fail to rank cases such as the one illustrated in Table 3, distribution F, even if it is apparent that nobody is worse-off? Reference has to be made to the fact that the Lorenz curves capture the "shares of income", not income units.
- ✓ **How is the "with policy" income distribution generated?** Selected trainees who are not familiar with how to build policy scenarios may not understand how, in practical cases, the "with policy" income distribution is generated, i.e. how to logically link the policy proposal to the new income distribution. In addition, the possibility to prepare and run exercises slightly more complex than the examples provided in the module with real data, has to be considered.

7.3 Complementary capacity building materials

This module is complemented by a set of ***slides*** which support the delivery of training lectures. The trainer may also consider presenting the relevant segment of the Armenia country case study based on real data (see reference below).

7.4 EASYPol links

This module belongs to a set of modules which discuss how to provide normative prescriptions when confronting alternative income distributions, i.e. how to identify the best income distribution in terms of social welfare, in a set of alternative income distributions. It is one of the modules composing a training path addressing **Analysis and monitoring of socio-economic impacts of policies.**

The following EASYPOL modules form a set of materials logically preceding the current module, which can be used to strengthen the background of the user:

- EASYPol Module 000, ***Charting Income Inequality: The Lorenz Curve.***
- EASYPol Module 001, ***Social Welfare Analysis of Income Distributions: Ranking Income Distributions with Lorenz Curves.***

Issues addressed in this module are further expanded in the following module:

- EASYPol Module 003, ***Social Welfare Analysis of Income Distributions: Ranking Income Distribution with Crossing Generalised Lorenz Curves***.

A case study presenting the use of Lorenz curves to rank income distributions in the context of an agricultural policy impact simulation exercise with real data is reported in the EASYPOL Module 042 ***Inequality and Poverty Impacts of Selected Agricultural Policies: The Case of Paraguay.***

8 FURTHER READINGS

Anand S., 1983. *Inequality and Poverty in Malaysia*, Oxford University Press, London, UK.

Lambert P. J., Aronson J. R., 1993. Inequality Decomposition Analysis and the Gini Coefficient Revisited, *Economic Journal*, 103, pp. 1221-1227.

Lambert P., 1993. *The Distribution and Redistribution of Income – A Mathematical Analysis*, Manchester University Press, UK, 2nd edition,

Lerman , Yitzhaki S., 1995. Changing Ranks and the Inequality Impacts of Taxes and Transfers, *National Tax Journal*, 48, pp. 45-59.

Sen A., 1997. *On Economic Inequality*, Oxford University Press, Oxford, UK, 2nd edition.