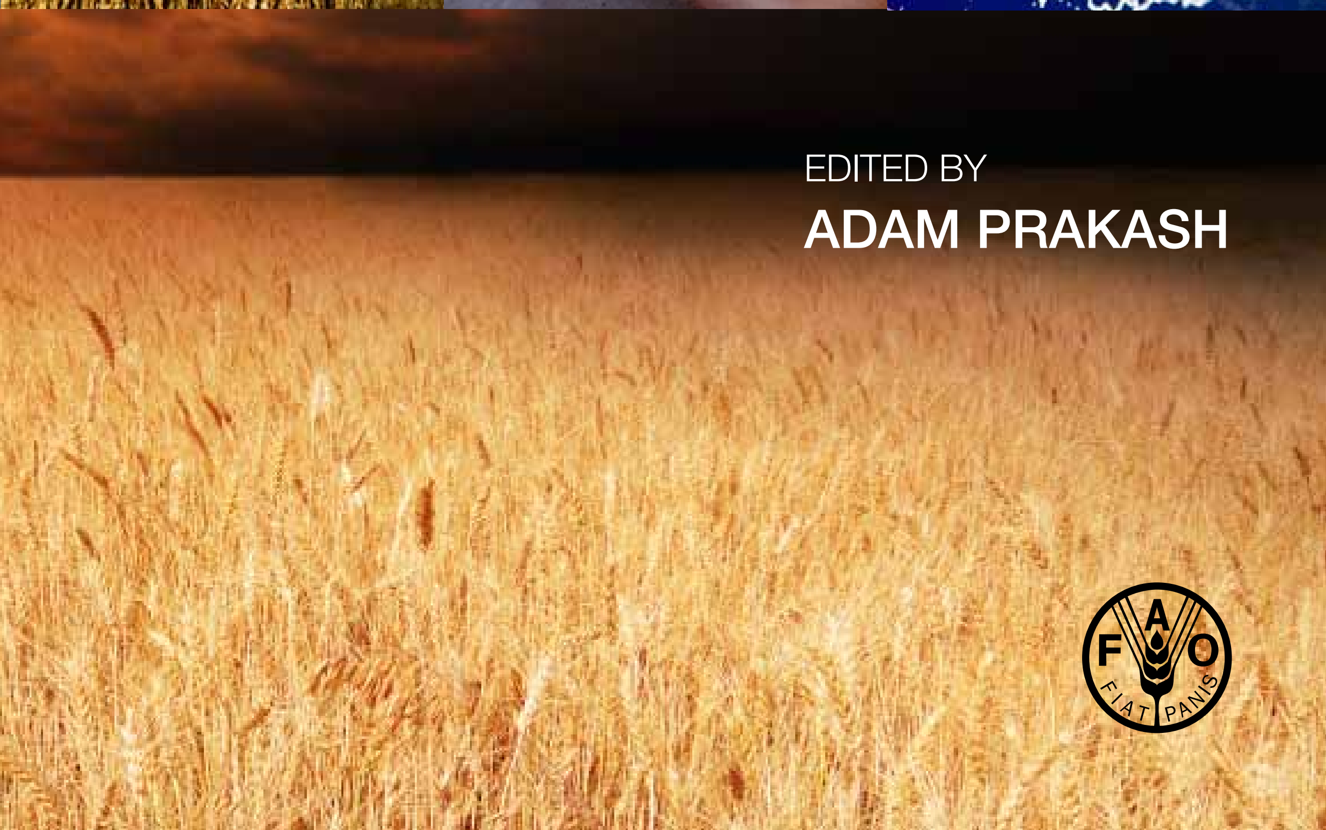
SAFEGUARDING FOOD SECURITY IN VOLATILE GLOBAL MARKETS



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Safeguarding food security in volatile global markets

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Contents

Chapter 15

Storage arbitrage and commodity price volatility

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This chapter is concerned with the role of storage arbitrage in shaping price dynamics. We apply the Pseudo Maximum Likelihood estimator of Deaton & Laroque (1995), corrected for numerical accuracy of the solution function and modified to allow for free disposal of excess stocks, to current series of annual average commodity prices. We confirm the fundamental result of Cafiero et al. (forthcoming) that the standard storage model in the tradition of Gustafson (1958) is indeed capable of explaining the most prominent features of the dynamics of commodity prices, including episodes of isolated price spikes and conditional high price volatility. Using a series of de-trended annual average prices for wheat, we demonstrate how to use the estimated model to generate distributions of the price next period, given the past history of prices and conditional on the amount of stocks implied by the current price. Such information should prove very useful in anticipating periods of high price volatility and in planning policies to prevent turmoil and crises in food markets.

Introduction

Understanding the dynamics of storable commodity prices and how they relate to fundamentals of supply and demand remains a challenge for policy analysts and economists. Episodes of sudden increase in price volatility, not necessarily aligned with detectable contemporaneous shocks in the underlying supply or demand, have significantly challenged the ability of economists to explain price dynamics. The occurrence of food price spikes, in particular, raises concerns for the welfare of the poor and most vulnerable, for whom even short periods of high food prices could have disrupting long-run consequences. Often, irrational speculative-driven bubbles have been blamed for such episodes of high price volatility, with no clear implications in terms of which possible policies could effectively prevent repetition of food price crises.

In this chapter we explore the role intertemporal arbitrage through storage plays in shaping the dynamics of prices by analysing a number of series of international agricultural

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commodity prices, extending to the latest available figures of 2010. Our empirical estimations confirm the practical relevance of a fundamental result of Cafiero et al. (forthcoming), namely, that storage arbitrage is capable of explaining the most prominent features of the dynamics of commodity price, including episodes of isolated price spikes and high price volatility. We show how to use the estimated model to generate expectations of future price given the past history of prices and conditional on the amount of current stocks, a predictive tool that should prove very useful in anticipating possible periods of high price volatility and as an aid in planning policies to prevent food crises.

Moving from the seminal contribution of Gustafson (1958), and the later ones of Samuelson (1971) and of Scheinkman & Schechtman (1983), we consider what is by now referred to as the "standard storage model" (Wright & Williams, 1982; Deaton & Laroque, 1992). An insight provided by this model is that storage arbitrage may induce serial correlation in prices even if the fundamental shocks driving production and demand are uncorrelated. The model also implies that the distribution of commodity prices is skewed even under symmetric supply and demand shocks. Under the truth of the model, time series of prices will show isolated spikes, a feature determined by the possibility of occurrence of stockouts, i.e. periods in which discretionary stocks (that is, quantities held in addition to minimum operational stocks) fall to zero. Storage, in fact, introduces a key nonlinearity in the market demand, implying two different regimes of price volatility, one in which abundant reserves can buffer the effects of negative shocks in supply, and another in which, at high prices, the low levels of stocks leave the market particularly vulnerable to shocks in supply or demand.

The theoretical soundness of the standard storage model has long been recognized. After the pioneering work of Gustafson (1958), who showed how to solve for the stochastic dynamic equilibrium of the model, and thus anticipating the rational expectation hypothesis of Muth (1961), Scheinkman & Schechtman (1983) extended the model to include supply that is responsive to economic incentives and identified the analytical relationship of competitive market prices and stocks with those of a social surplus problem with possibly non-stationary fundamentals. Williams & Wright (1991) make a thorough exploration of the economic implications of the model and extend it in many important directions, including the implications of non-competitive behaviour and the effects of government interventions and speculative attacks. Bobenrieth H. et al. (2010) offer a rational explanation for episodes of high volatility in price series: in a storage model with continuously positive stocks, they prove the existence of rational price bubbles in expectation and show the possibility of occurrence of arbitrarily long periods with prices increasing at rates faster than the discount rate. They also prove that there exists a state-dependent finite horizon beyond which the price realization stays below any arbitrary fraction of the corresponding profile of conditional expectations. Their result raises delicate questions regarding the effects of speculation and of the standard interpretation of futures prices as conditional expectations.

The empirical validity of the standard storage model, however, has been repeatedly questioned and a clear consensus is yet to be reached, as highlighted recently by Deaton: "We have a long-established theory–whose insights are deep enough that *some* part of them *must* be correct–which is wildly at odds with the evidence, and where it is far from obvious what is wrong, or how the theory might be amended to give us a better handle on the mechanisms at work." (Deaton, forthcoming, emphasis in the original)

Empirical tests of the storage model began with Deaton & Laroque (1992) who introduced a Generalized Method of Moments (GMM) estimator and showed that the dynamics of the thirteen commodities they analysed are consistent with a two-regime autoregressive process in which current price is correlated to the previous one only when the latter is below a given threshold (a feature that *may* be due to storage). The results were encouraging, though they could not be deemed as definitive evidence in support of the storage model, given that other competing models of price behaviour could generate the two-regime auto-regression used to construct the GMM estimator (see Chapter 2). Deaton & Laroque supported their reservations, in spite of the GMM results, by simulations of versions of the model with different sets of parameter values which generate first order autocorrelations far lower than those observed on the annual time series of prices for thirteen commodities they consider. Given the limited range of parameter values they explored, however, they left open the question of whether there are other plausible values for the fundamental parameters of the model that could generate higher levels of serial correlation that would match those observed in the series of actual prices.

A few years later, introducing a path-breaking Pseudo Maximum Likelihood (PML) empirical estimation approach, Deaton & Laroque (1995, 1996) were able to estimate the parameters of the underlying demand and storage cost relations for the same thirteen series of annual average commodity prices they had considered previously, thus allowing for a formal test of a simplified version of the model, assuming linear demand, and storage cost due to proportional decay of the amount stored. They could not confirm the positive results of the GMM estimation, and found that, in order to account for observed levels of autocorrelation in their model, it was necessary to relax the assumption of i.i.d. harvest. When they did so, they were "forced to attribute effectively all [of the autocorrelation] to the underlying [harvest] process."(Deaton & Laroque, 1996, p.921) Their conclusion is summarized in Deaton & Laroque (2003, p.2): "[T]he speculative model, although capable of introducing some autocorrelation into an otherwise i.i.d. process, appears to be incapable of generating the high degree of serial correlation of most commodity prices."

The strong emphasis they put on the model rejection is surprising, considering that it would be too much to expect to be able to explain the evolution of prices of a diverse set of major commodities over almost a century with a model based on stationary linear consumption demand, a fixed distribution of market disturbances, a constant interest rate, and no supply response. Indeed, it would be very surprising if long-run market influences such as trends in the yields or changes in the structure of the supply and demand shocks had no detectable effect on the evolution of prices, let alone the presence of various forms of market intervention that have characterized the commodities analysed. The failure that Deaton & Laroque (1995, 1996) found is best interpreted as a rejection of the particular specification they adopted, not of the role of speculative storage in general. Also, it should be considered that practical implementation of the dynamic equilibrium model implied by the theory of storage arbitrage is *very* difficult. In Angus Deaton's own words, in this type of work "[I]t is difficult to disentangle the auxiliary assumptions from the central core that we want to test, the computations are time-consuming and error-prone, the substance of the problem tends to be lost in the sometimes byzantine complexity and programming of the estimation, and it is hard to get a sense of why the results are what they are." Deaton (2010, p.8)

Considering such difficulties, and being puzzled by the strong inconsistency between the results of the PML estimations and the previous GMM results on the very same price data series, Cafiero (2002) explored the possibility that the results in Deaton & Laroque (1995, 1996) might have been conditioned by some computational or programming problem. In fact, he was able to closely replicate the PML estimation of Deaton & Laroque (1995, 1996), and found that they were extremely sensitive to an implementation choice, namely the degree of accuracy in the numerical approximation of the equilibrium function. Even slight increases in the number of grid points used for function approximation led to dramatic changes of the estimated values. After improving on that particular aspect of the numerical implementation, and by exploring alternative assumptions on the nature of storage costs, Cafiero et al. (forthcoming) reach more positive conclusions regarding the empirical relevance of the standard storage model.

The results in Cafiero et al. (forthcoming) allow for the separation of two conceptually distinct issues. One is whether the standard storage model, even in a very simple version with linear demand, constant interest rate, and i.i.d. net supply shocks, is at all capable of reproducing the high levels of serial correlation observed in actual data series of commodity prices. The other one concerns the ability of estimated storage models to fit observed data from the point of view of a wider set of indicators. In Cafiero et al. (forthcoming) we tackle each of the issues in turn. First, we simulate the storage model for different values of the parameters and discuss the implied key characteristics of the price series. In contrast to the simulation presented by Deaton & Laroque (1992), we take a much steeper consumption demand, reflecting the fact that consumption of basic commodities is notably price inelastic. We find that, with the low price-response consumption demand, storage plays a greater role in determining price behaviour, causing a dramatic increase in the correlation measured on long series of simulated prices. Then, using the PML empirical estimation method and the same samples of commodity prices of Deaton & Laroque (1995, 1996), but improving on the accuracy of the numerical solution of the model, we find that the estimated model is indeed capable of replicating central features observed of real prices, including serial correlation, coefficient of variation, skewness and kurtosis for several of the commodities they considered.

In this chapter we elaborate further on the issues discussed in Cafiero et al. (forthcoming) by extending the analysis in two directions. First, as noted in Cafiero et al. (forthcoming), the assumptions of constant, positive marginal storage cost coupled with linear consumption demand can generate negative prices in this model. To avoid this problem, the model of Deaton & Laroque (1992) needs to be modified to allow for free disposal of the excess amount stored, and thus we need to introduce a more general version of the results of Theorem 1 and Theorem 3 of Deaton & Laroque (1992), and a proof of existence of a unique invariant distribution for prices. We also present a proof of the identification result (see the Proposition in Deaton & Laroque 1996) appropriate for our model.

Then, we apply the model to series of annual average prices of maize, wheat, rice and sugar, covering the period 1900-2010. We also fit the model to shorter series of the same prices, covering the period 1949-2010, and which we de-trend in consideration that our theoretical model is based on stationary prices.

Our results confirm in general the ability, already demonstrated by earlier findings, of the standard storage model to replicate the central features of the dynamics of prices for storable commodities. This means that the role that storage plays in determining price behaviour of the commodity prices should not be neglected. Further, we demonstrate how the models we estimate can be used to form predictions of future prices, and discuss the implication that such information should have for analyses of food security in general, and to inform policy-makers in particular with respect to the possibility of anticipating food price crises.

The remainder of the chapter is organized as follows. In the next section, we present the theoretical model and the main theoretical results on existence and uniqueness of the Stationary Rational Expectation Equilibrium for the model that allows for free disposal of the excess stocks, relegating to an appendix the formal proof of the various theorems and propositions introduced. In the following sections we describe, in turn, the estimation procedure and how we implement it, the data used, and the results obtained. A final section with discussion of the implications of our results for policy analysis and suggestions for further research closes the chapter.

A model with constant marginal storage cost and free disposal

We model a competitive commodity market in which, in any given period, available supply is comprised of a random component (the "harvest") plus the amount of stocks carried in from the previous period. Available supply is partly used for consumption and partly purchased by speculators motivated by expected profits. Speculators are assumed to form rational expectations of the next period price.

Supply shocks, ω_t, are i.i.d., with support in $\mathbb{R}$ that has lower bound $\underline{\omega} \in \mathbb{R}$. Storers are risk neutral and face a constant discount rate $r > 0$. Stocks physically deteriorate at rate d, with $0 \leq d < 1$, and the cost of storing $x_t \geq 0$ units from time t to time $t+1$, paid at time t, is given by kx_t, with $k > 0$.

One possible state variable for this model is z_t, the total available supply at time t, defined as $z_t \equiv \omega_t + (1-d)x_{t-1}$, with $z_t \in Z \equiv [\underline{\omega}, \infty[$. Price is formed as $p_t = F(c_t)$, where consumption $c_t \equiv z_t - x_t$. The inverse consumer demand, $F : \mathbb{R} \to \mathbb{R}$, is continuous, strictly decreasing, and have the following properties: $\{c : F(c) = 0\} \neq \emptyset$, $\lim_{c \to -\infty} F(c) = \infty$, and $\left(\frac{1-d}{1+r}\right) EF(\omega_t) - k > 0$, where E denotes the expectation taken with respect to the random variable ω_t.

A stationary rational expectations equilibrium (SREE) in this model is a price function $p : Z \to \mathbb{R}$ which describes the current price p_t as a function of the state z_t, and which satisfies, for all z_t,

$$p_t = p(z_t) = \max\left\{\left(\frac{1-d}{1+r}\right) E_t p\Big(\omega_{t+1} + (1-d)x_t\Big) - k,\ F(z_t)\right\} \tag{1}$$

where:

$$x_t = \begin{cases} z_t - F^{-1}\Big(p(z_t)\Big), & \text{if} \quad z_t < z^* \equiv \inf\{z : p(z) = 0\} \\ z^* - F^{-1}(0), & \text{if} \quad z_t \geq z^*. \end{cases} \tag{2}$$

As the ω_t's are i.i.d., p is the solution to the following functional equation:

$$p(z) = \max\left\{\left(\frac{1-d}{1+r}\right) Ep\Big(\omega + (1-d)x(z)\Big) - k, F(z)\right\}, \tag{3}$$

and

$$x(z) = \begin{cases} z - F^{-1}\Big(p(z)\Big), & \text{if} \quad z < z^* \\ z^* - F^{-1}(0), & \text{if} \quad z \geq z^*. \end{cases} \tag{4}$$

Existence and uniqueness of the SREE, $p(z)$, as well as some of its properties are given by the following Theorem:

Theorem 1. *There is a unique stationary rational expectations equilibrium p in the class of continuous non-increasing functions. Furthermore, if* $p^* \equiv \left(\frac{1-d}{1+r}\right) Ep(\omega) - k$, *then:*

$$\begin{array}{lll} p(z) = F(z), & \textit{for} & z \leq F^{-1}(p^*), \\ p(z) > \max\{F(z), 0\}, & \textit{for} & F^{-1}(p^*) < z < z^*, \\ p(z) = 0, & \textit{for} & z \geq z^*. \end{array}$$

p is strictly decreasing whenever it is strictly positive. The equilibrium level of inventories, $x(z)$, is strictly increasing for z in $[F^{-1}(p^*), z^*]$.

The following comparative statics result parallels Theorem 3 of Deaton & Laroque (1992).

Theorem 2. *The equilibrium price function p, the associated cut-off price p^* and the inventory demand $x(z)$ are non-decreasing in the discount factor $\beta \equiv 1/(1+r)$. They are non-increasing in the marginal storage cost k, and they do not increase when there is a first-order stochastic increase in the distribution of supply shocks ω_t. Moreover, if $F(c)$ is convex, then $p(z)$ is convex, and both $p(z)$ and $x(z)$ do not decrease when the distribution of supply shocks is modified through a mean-preserving spread.*

We now establish existence and uniqueness of the invariant distribution for prices.

Theorem 3. *Let $\omega_t \in [\underline{\omega}, \overline{\omega}], -\infty < \underline{\omega} \leq \overline{\omega} < +\infty$. Suppose that ω_t has a mixed discrete-continuous distribution of the form $\alpha L_d + (1-\alpha)L_c$, where $0 \leq \alpha \leq 1$, L_d is a discrete distribution, and L_c is an absolutely continuous distribution with continuous and positive derivative m on $[\underline{\omega}, \overline{\omega}]$. Then the Markov process of available supply has a unique invariant distribution which is a global attractor, and has no atoms. Furthermore, the Markov process of prices has a unique invariant distribution, which is a global attractor.*

Remark. *Suppose that the inverse consumption demand $F : [0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ is continuous, strictly decreasing, and satisfies $F(0) = +\infty$, $\{z : F(z) = 0\} \neq \emptyset$. If, under the assumptions of Theorem 3, $\underline{\omega} = 0$, $\omega_t = 0$ with positive probability, then:*

- *The invariant distribution of the price process has infinite expectation.*
- *Given any initial finite price p_0, the conditional expectation and the conditional variance of prices p_t diverge to $+\infty$, as $t \to +\infty$.*
- *Although the conditional expectation of prices increases with no bound, the probability that the sequence of realized prices stays below its own conditional expectation can be made arbitrarily close to one, by choosing a far enough finite time horizon.*

The following Proposition parallels Proposition 1 in Deaton & Laroque (1996, p. 906). This result allows identification of the model when only prices are observed, by arbitrarily setting the mean and the standard deviation of the supply shocks ω_t to be zero and one, respectively.

Proposition. *Consider a model with discount rate r, stocks deterioration parameter d, constant marginal and average storage cost k, supply shocks ω_t, and inverse demand function F. Any other model with discount rate r, stocks deterioration parameter d, constant marginal and average storage cost k, supply shocks $\widetilde{\omega}_t \equiv \sigma\omega_t + \mu$, and inverse demand function $\widetilde{F}$ satisfying $\widetilde{F}(\sigma z + \mu) = F(z)$, has the same rational expectations price process as the base model.*

The proof of all the Theorems, the Remark and the Proposition are presented in the Appendix.

Estimation

We estimate the model described above, assuming a linear inverse demand function, $F(c) = a + bc$, with $b < 0$, using the Pseudo Likelihood Maximization procedure introduced by Deaton & Laroque (1995, 1996). Such a procedure is based on the assumption that prices, conditional on the previous period price, $p_{t+1}|p_t$, are normally distributed, so that the log-pseudo likelihood function can be formed as follows:

$$\ln L = \sum_{t=1}^{T-1} \ln l_t = 0.5\left[-(T-1)\ln(2\pi) - \sum_{t=1}^{T-1} \ln s(p_t) - \sum_{t=1}^{T-1} \frac{\left(p_{t+1} - m(p_t)\right)^2}{s(p_t)} \right], \tag{5}$$

where $m(p_t)$ and $s(p_t)$ are the first and second central moments of the conditional price, respectively. Note that the distribution of prices in this model cannot be expected to be normal even if the underlying supply shocks are, because of the nonlinearity of the function that maps harvests to prices.

As described in Deaton & Laroque (1995), evaluation of the conditional expectation $m(p_t)$ and the conditional variance $s(p_t)$ is conducted in several steps, each of which in turn requires some practical assumptions, whose impact on the overall reliability of the implemented procedure and on the properties of the estimator is not obvious. The most delicate assumptions concern: 1) the choice of the approximation scheme for the SREE function p which, under the parameterizations maintained in this chapter, will have no closed form; and, 2) the way in which the expectations are calculated. Though various numerical approximation schemes are possible, including low order polynomials (Williams & Wright, 1991) or orthogonal collocation methods based on Chebychev polynomials (Miranda, 1985), we follow Deaton & Laroque (1995) and use cubic splines over a grid of equally spaced points. As in Cafiero (2002) and in Cafiero et al. (forthcoming), we use a large number of grid nodes to ensure sufficiently close approximation of the p function in the vicinity of the kink point corresponding to the threshold price p^*.

To take expectations with respect to the random shock ω, we substitute the integral over a continuous range of values of the harvest and corresponding density with a summation over a discrete set of 10 unequally spaced nodes ω^n and corresponding weights π^n.[2] Using this approximation, condition (1) can be expressed as:

$$p_t = p(z_t) = \max\left\{ \left(\frac{1-d}{1+r}\right) \sum_{n=1}^{N} p\left(\omega^n + (1-d)x_t\right)\pi^n - k,\ a + bz_t \right\}. \tag{6}$$

Expression (6) defines a functional that can be solved iteratively for the function p. As noted above, we approximate p by a cubic spline over a grid of sufficiently many points to give a good approximation around the kink point and given a suitable range of values for z_t.[3]

Using the approximate SREE price function p, we calculate the first two moments of p_{t+1} conditional on p_t as:

$$m(p_t) = \begin{cases} \sum_{n=1}^{N} p\left(\omega^n + (1-d)\left(p^{-1}(p_t) - F^{-1}(p_t)\right)\right)\pi^n, & \text{if} \quad p_t > 0 \\ \sum_{n=1}^{N} p\left(\omega^n + (1-d)\left(z^* - F^{-1}(0)\right)\right)\pi^n, & \text{if} \quad p_t = 0, \end{cases}$$

[2] Deaton & Laroque (1995, section 3.1) chose a discretization with ten equally probable values. Our choice of nodes and weights is based on a Gauss-Hermite quadrature scheme optimally designed to approximate expectations of functions of standard normal random variables (see Judd, 1998, Section 7.2.).

[3] Our Matlab routines, available upon request, are based on the algorithm sketched in Deaton & Laroque (1995, Section 3.2), and suitably modified to reflect our extensions of the theoretical model, to include the restrictions represented by (2).

and

$$s(p_t) = \begin{cases} \sum_{n=1}^{N} p(\omega^n + (1-d)[p^{-1}(p_t) - F^{-1}(p_t)])^2 \pi^n - m^2(p_t), & \text{if} \quad p_t > 0 \\ \sum_{n=1}^{N} p(\omega^n + (1-d)[z^* - F^{-1}(0)])^2 \pi^n - m^2(p_t), & \text{if} \quad p_t = 0. \end{cases}$$

that we use to evaluate the expression in (5), which becomes a function of the parameters of the model, $\{a, b, k, d, r\}$.

Even though (5) is not the true log log-likelihood (in presence of storage prices will not be distributed normally), by the arguments in Gourieroux et al. (1984) the estimates are consistent.[4]

Data

Our data consists of annual prices for maize, wheat, rice and sugar extending over the period 1900-2010. The time series are presented in Pfaffenzeller et al. (2007) for the period up to 2003. We extend them from 2003 to 2010 using price data provided by the World Bank Development Prospects Group, following the procedure suggested therein to ensure homogeneity of the series. The monthly figures reported in the "World Bank Pink Sheets" are averaged over the calendar year, and then divided by the respective 1977-79 average. The normalized nominal values are then divided by the annual United States Consumer Price Index as reported by the US Bureau of Labor Statistics, to produce the real price indices depicted in Figure 15.1.

Results and discussion

We estimate the presented model using maize, wheat, rice and sugar price data. Although in principle the interest rate can be treated as a parameter, we do not attempt to estimate it, rather we set it at 2 percent, close to estimates of the real risk-free cost of capital in the United States and the United Kingdom in the twentieth century. (See for example Campbell, 1999; Goetzmann & Ibbotson, 2005; Shiller, 2005). Also, in all estimations the parameter d was never significantly different from zero, and therefore we set it at zero and estimate only the three parameters a, b and k in the results we present.

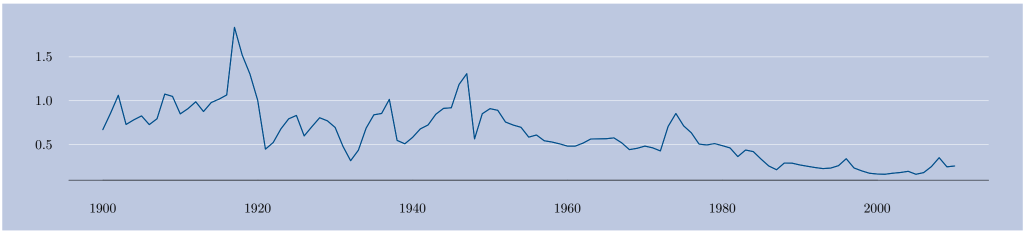
We start by using the series of deflated prices extending from 1900 to 2010, and are able to identify a well-behaved maximum of the Pseudo Likelihood function for Maize and Sugar.[5] The estimated parameters are reported in Table 15.1, along with the value of the maximized pseudo likelihood, the implied threshold price, p^*, and the resulting number of stockouts over the sample period. These estimates are largely in line with the results presented in Cafiero et al. (forthcoming, Table 7) and imply that there have been no stockouts for maize, and 22 stockouts for sugar over the period.

[4] Implementing a proper maximum likelihood estimator for this model, based on price observation but assuming a distribution for the harvests, poses significant challenges if stockouts are allowed. Forming the likelihood function would require the Jacobian of a non-differentiable function, implying discontinuities in the function to be maximized. Miranda & Rui (1999) present a maximum likelihood estimator based on the distribution of harvests, but in their model discontinuity of the objective function is eliminated by ruling out stockouts.

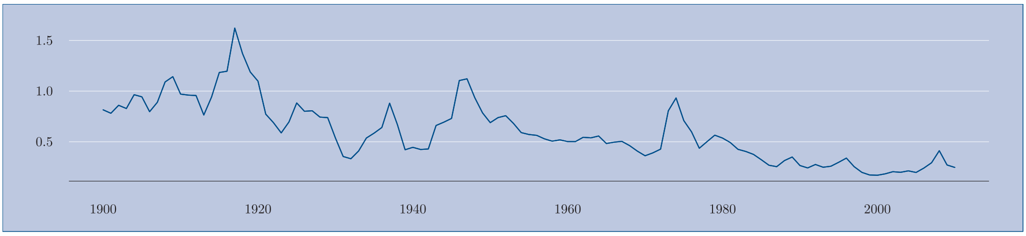
[5] For the other two series, the estimator did not converge, tending towards estimating an infinitely steep inverse demand function.

Figure 15.1: Annual real prices of maize, wheat, rice and sugar: 1900-2010

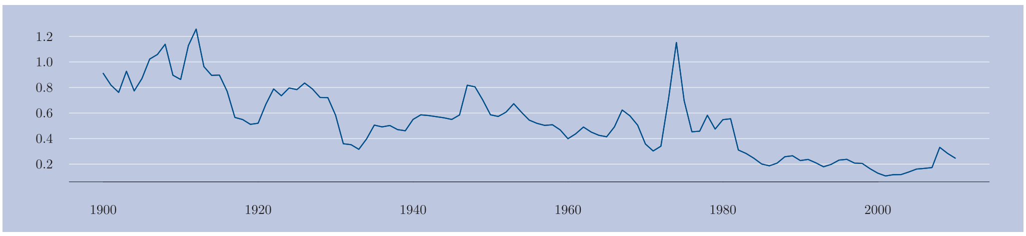
(a) Maize



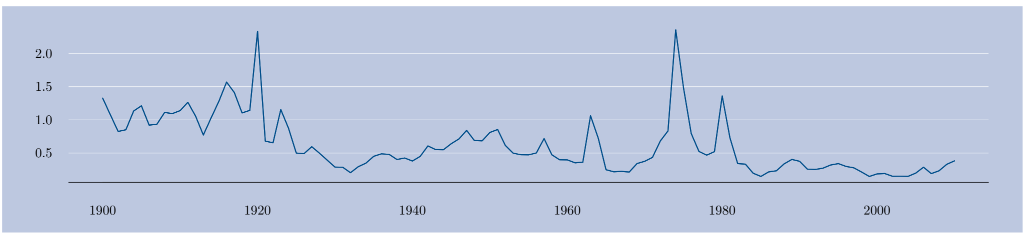
(b) Wheat



(c) Rice



(d) Sugar



Source: Pfaffenzeller et al. (2007) and The WorldBank Prospects Group.

One of the purposes of the estimations we present is to assess the ability of the storage model to capture key features of the distribution of prices. The model implies a long-run, invariant distribution of prices, whose moments could be compared with those observed in the data, as it is done for example in Deaton & Laroque (1996). This, however, would provide only a partial assessment of the actual fit of the model, given that observed series of prices are always too short to reveal higher-order moments of the underlying price distribution,

Table 15.1: Parameter estimates on the series of deflated price data: 1949-2010

Commodity	Parameters⋆			PL	p^*	No. stockouts
	a	b	k			
Maize	0.7427	-3.3805	0.0054	82.5341	2.6090	0
	(0.185)	(0.691)	(0.0042)			
Sugar	0.4606	-1.0162	0.0192	24.5539	0.9272	22
	(0.1359)	(0.1788)	(0.0120)			

⋆ d is fixed at 0 and r is fixed at 0.02. Asymptotic standard errors in parentheses.

Table 15.2: Predicted features of price distributions: 1900-2010

Commodity	Moments⋆						% stockouts	Prob (s>0)
	μ	ρ_1	ρ_2	CV	Skewness	Kurtosis		
Maize	35.880	57.315	51.251	31.257	18.516	32.341	1.09	0.4381
Sugar	95.540	68.514	57.472	60.417	24.300	21.961	5.19	0.9399

⋆ Each number is the percentile of the distribution corresponding to the observed value of each given moment. μ is the mean, ρ_1 is the first order serial correlation, ρ_2 is the second order serial correlation, CV is the coefficient of variation.

Note: For each commodity, we generated a series of 300 000 prices using the estimated parameters reported in Table 15.1. We then extracted all possible subsamples of the same length of the data series, and measured mean, first order and second order autocorrelation, coefficient of variation, skewness and kurtosis on each simulated sample. The numbers in the table are the percentiles of these simulated distributions corresponding to the values observed in the data. Prob(s>0) is the relative frequency of samples with at least one stockouts in the simulated series.

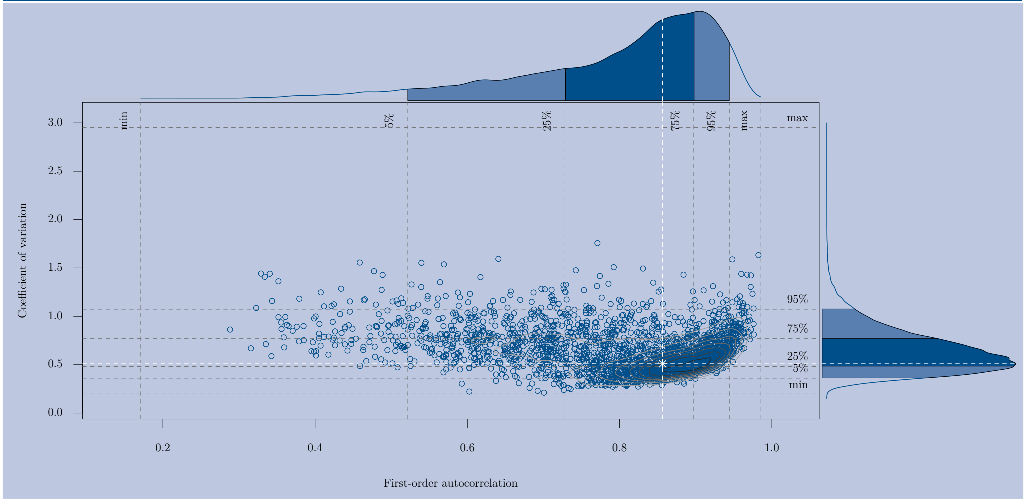
especially if such a distribution is expected to be highly skewed. On the other hand, to be confident that a model is indeed capable of replicating such higher-order moments is crucial if the model is to be used to predict such features as price spikes, runs of low prices, etc.

To conduct a more coherent assessment of the models' fit, we follow the method presented in Cafiero et al. (forthcoming) and use the estimated parameters to simulate a long series of artificial prices[6] and then extract from it all possible subsamples of the same length as the observed data. On each subsample we measure the mean, the first and second order serial correlation, the coefficient of variation, the skewness and the kurtosis, in addition to calculating the number of stockouts. Finally, we identify in each of the simulated distribution of small sample price moments, the percentiles corresponding to the values of the corresponding moments as observed in the price data. Table 15.2 shows the values of the percentiles in the simulated distributions corresponding to the mean, the first and second order serial correlation, the coefficient of variation, the skewness and the kurtosis, observed on the price series for maize and sugar. With the exception of the mean for sugar, all observed price moments lie well within symmetric 90 percent confidence regions.

We can also consider the possible correlation between different features of the dynamics of prices to assess whether the model can generate series that have, for example, levels of

6 In this chapter we simulate series of length 300 000.

Figure 15.2: Empirical joint distribution of first-order correlation & coefficient of variation as predicted by the storage model in samples of length 111, and estimated on maize deflated prices: 1900-2010



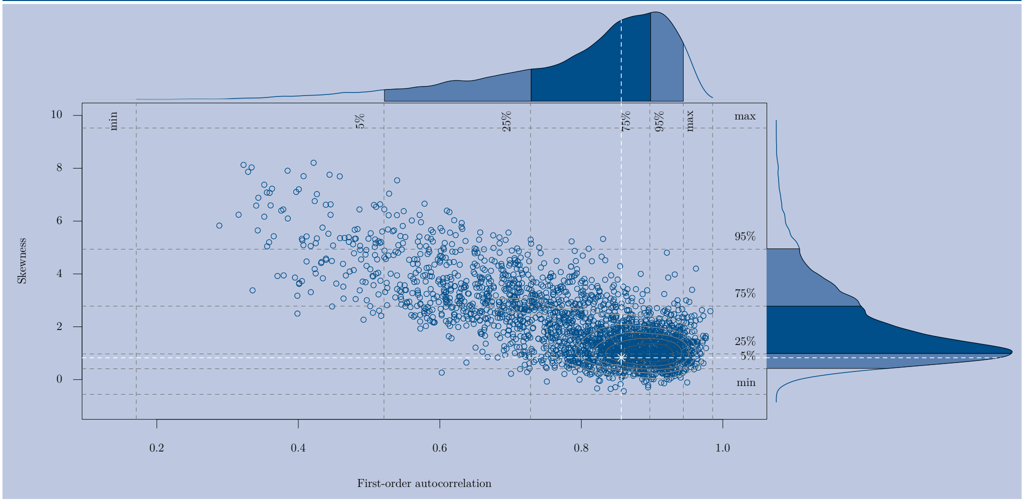
Note: The shaded areas in the densities and the corresponding grid lines in the plots indicate the minimum, the 5th, 25th, 75th and 95th percentiles and the maximum value obtained in the simulation. A white star indicates the position of the the first order serial correlation and coefficient of variation measured on the series of maize prices.

both correlation and skewness similar to those observed in the data. The graph in Figure 15.2 illustrates the scatter plot of the values of coefficient of variation and of first order correlation measured on each simulated series, along with the marginal densities of the two variables, while the graph in Figure 15.3 does the same for first order serial correlation and skewness, both for the case of maize. The results of a similar exercise for sugar are reported in the graphs of Figure 15.4 and 15.5. The location of the combination of the corresponding two parameters measured on the series of data used to estimate the model is indicated by a white star in each of the graphs.

The graphs confirm that series of prices very much like those observed for maize and for sugar over a period of more than a century could well be generated by the standard storage model, even under an admittedly very simplified specification, using normally distributed i.i.d. supply shocks, linear demand and constant real interest rate. To some extent, this is a surprising result, considering that no one would defend the truth of the many maintained assumptions, such as, for example, the fact that agricultural commodity prices have been stationary over such a long period of time. Even a simple visual inspection of the series plotted in Figure 15.1 reveals that all price series show an obvious downward trend, and, possibly, a structural change after World War II. Indeed, the presence of such a trend poses a challenge to a model that assumes that the fundamentals are stationary. In fact, this may well be the reason that the storage model, as specified, cannot be fitted to the other two series of prices we considered.

In light of this, in the next set of estimations we consider the de-trended series covering the period 1949-2010, obtained by subtracting from the original series a log-linear trend

Figure 15.3: Empirical joint distribution of first-order correlation & skewness as predicted by the storage model in samples of length 111, and estimated on maize deflated prices: 1900-2010



Note: The shaded areas in the densities and the corresponding grid lines in the plots indicate the minimum, the 5th, 25th, 75th and 95th percentiles and the maximum value obtained in the simulation. A white star indicates the position of the the first order serial correlation and skewness measured on the series of maize prices.

Table 15.3: Parameter estimates on the series of de-trended data: 1949-2010

Commodity	Parameters⋆			PL †	p^*	No. stockouts
	a	*b*	*k*			
Maize	1.1280	-0.6819	0.0398	19.9838	1.3344	6
	(0.075)	(0.074)	(0.015)			
Wheat	1.0331	-0.7465	0.0232	19.4156	1.3146	8
	(0.002)	(0.001)	(0.0003)			
Rice	1.0359	-1.0046	0.0409	-1.7488	1.4082	9
	(0.004)	(0.002)	(0.0006)			
Sugar	1.1104	-2.2488	0.0654	-36.0000	2.0720	3
	(0.493)	(0.447)	(0.063)			

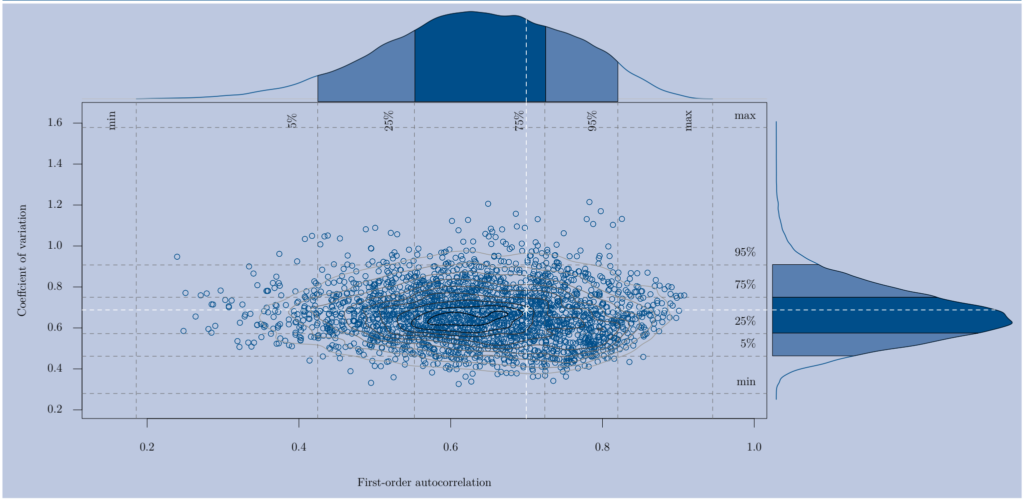
⋆ *d* is fixed at 0 and *r* is fixed at 0.02. Asymptotic standard errors in parentheses. † PL is the value of the maximized Pseudo Likelihood.

component estimated by ordinary Least Squares. The resulting de-trended series of real prices are depicted in Figure 15.6.

Not surprisingly, we are able to fit the storage model to all of the de-trended series. Table 15.3 reports the results of the estimations, and Table 15.4 the assessment of the ability of the estimated models to reproduce the moments of the price data.

With the exception of first order serial correlation for maize, all of the moments of the data series are within the central 90 percent confidence regions, demonstrating that the ability

Figure 15.4: Empirical joint distribution of first order correlation and coefficient of variation as predicted by the storage model in samples of length 111, and estimated on sugar deflated prices: 1900-2010



Note: The shaded areas in the densities and the corresponding grid lines in the plots indicate the minimum, the 5th, 25th, 75th and 95th percentiles and the maximum value obtained in the simulation. A white star indicates the position of the the first order serial correlation and coefficient of variation measured on the series of sugar prices.

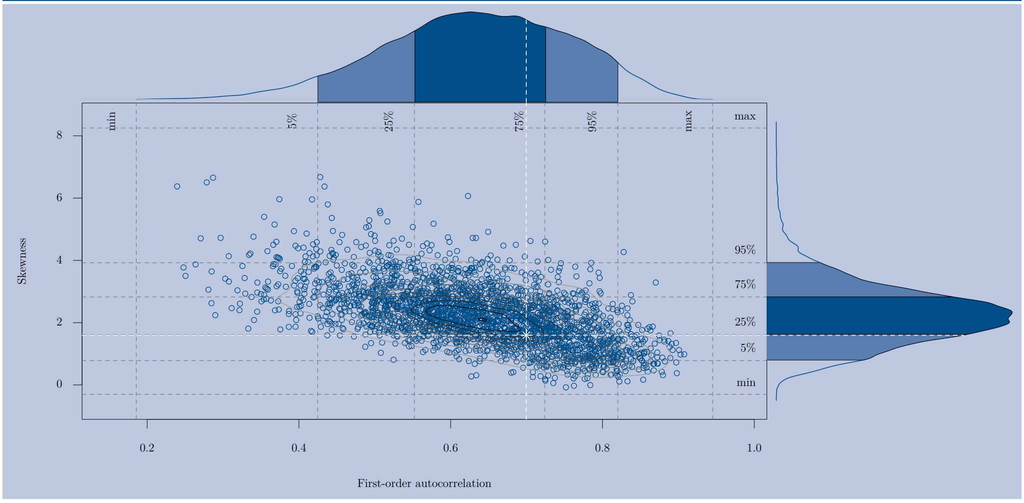
Table 15.4: Predicted features of price distributions. De-trended data: 1949-2010

Commodity	Moments*					
	μ	ρ_1	ρ_2	CV	Skewness	Kurtosis
Maize	15.898	99.036	90.399	43.562	25.646	23.257
Wheat	49.768	92.610	60.542	49.582	47.970	43.477
Rice	58.194	87.344	42.991	58.445	70.165	70.333
Sugar	57.898	50.023	17.705	68.264	91.855	89.533

* Each number is the percentile of the distribution corresponding to the observed value of each given moment. μ is the mean, ρ_1 is the first order serial correlation, ρ_2 is the second order serial correlation, CV is the coefficient of variation.
Note: For each commodity, we generated a series of 300 000 prices using the estimated parameters reported in table 15.1. We then extracted all possible subsamples of the same length of the data series, and measured mean, first order and second order autocorrelation, coefficient of variation, skewness and kurtosis on each simulated sample. The numbers in the table are the percentiles of these simulated distributions corresponding to the values observed in the data.

to fit many characteristics of the distribution of commodity prices must be recognized as a general feature of the standard storage model, despite the many simplifying assumptions, some of which –such as our choice of assuming a log-linear form of the trend– are admittedly ad hoc, and others –such as the assumption of a constant real interest rate over long periods, or the absence of policy distortions– are clearly at odds with what we know of the markets

Figure 15.5: Empirical joint distribution of first-order correlation & skewness as predicted by the storage model in samples of length 111, and estimated on sugar deflated prices: 1900-2010



Note: The shaded areas in the densities and the corresponding grid lines in the plots indicate the minimum, the 5th, 25th, 75th and 95th percentiles and the maximum value obtained in the simulation. A white star indicates the position of the the first order serial correlation and skewness measured on the series of sugar prices.

of these commodities over the past sixty years. We do not pursue this particular issue further in this chapter, postponing the exploration of how to relax the assumptions to future very promising research. What suffices here, is to have demonstrated once again that the standard storage model is empirically validated by a broad set of agricultural commodity price series, including those of two of the World's major food staples such as wheat and rice.

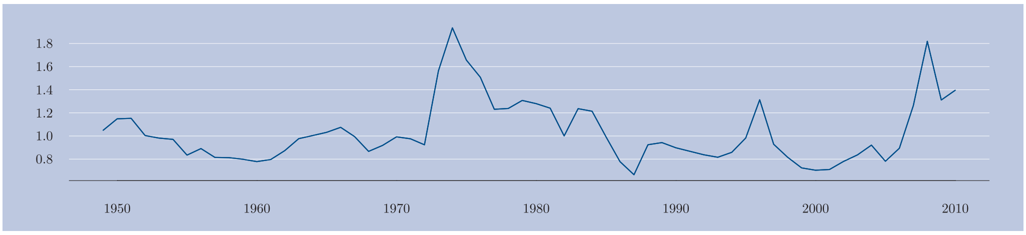
Our next objective is to demonstrate the potential usefulness of the model for policy analysis, and we base the discussion on one of the features of commodity price dynamics that has attracted much policy attention recently.

One important focus of the ongoing debate on commodity price volatility is the occurrence of price spikes and how they relate to the evolution of market fundamentals (production, trade, consumption and stock changes). The inability to closely match price spikes with current production shortfalls, for example, has led many to invoke irrational market behaviour, and to claim that uncontrolled speculation must be at the origin of what should be deemed to be commodity price bubbles. In the storage model, price spikes are predictable events, associated with periods in which discretionary stocks are at minimum levels. Indeed, in all price series we consider, the estimates imply spikes, rationalized here as periods of effective stockouts. For all commodities, the estimated models identifies a small number of such occurrence, as reported in last column of Table 15.3, corresponding to the peaks in the graphs of Figure 15.6.

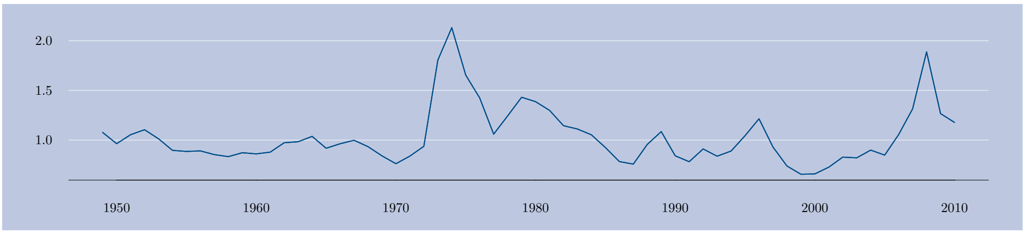
To explore the implications of the estimated models in terms of predicted number of stockouts we measure the frequency of stockouts in the long-run distribution of prices, and the relative frequency of samples with at least one, more than five, and more than ten

Figure 15.6: Annual de-trended real price indexes of maize, wheat, rice and sugar: 1949-2010

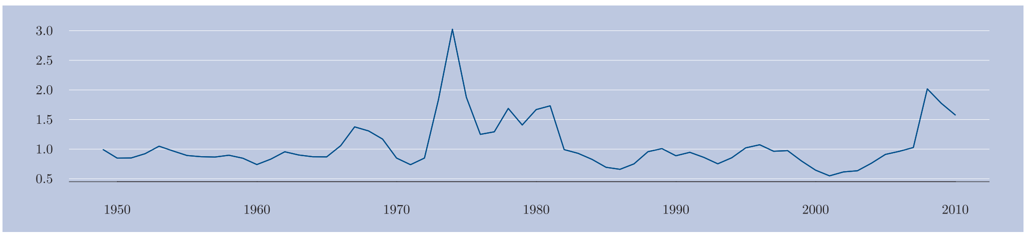
(a) Maize



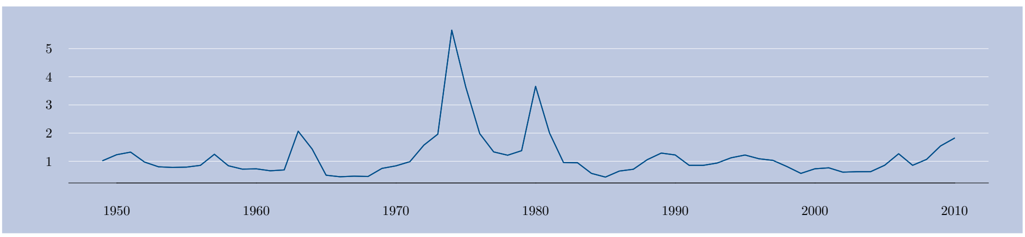
(b) Wheat



(c) Rice



(d) Sugar



Source: Pfaffenzeller et al. (2007) and WorldBank Prospect Groups.

stockouts, in the large number of simulated samples of length 62. Table 15.5 shows the results for the four different commodities.

As we can see, the estimated models imply that price spikes in the markets for these commodities are not so rare. Over a very long span of time, about 15 percent of the periods for maize, 10 percent for wheat and rice and 7 percent for sugar would correspond to a stockout. That at least one stockout would occur in a period of 62 years is very likely, with probability

Table 15.5: Predicted incidence of stockouts

Commodity	% stockouts	Prob($s > 0$)	Prob($s > 5$)	Prob($s > 10$)
Maize	14.89	0.9965	0.8019	0.3567
Wheat	10.06	0.9678	0.5327	0.1331
Rice	10.56	0.9746	0.5636	0.1523
Sugar	7.06	0.9020	0.3197	0.0511

Note: % stockouts is the percent of prices, over a long series of 300 000, exceeding the estimated threshold value p^* as reported in Table 15.3. s indicates the number of stockouts in a series of 62 prices. The probability of s exceeding n is estimated as the ratio between the number of samples containing more than n stockouts and the total number of simulated samples.

ranging from 0.9 for sugar to over 0.99 for maize. Even periods with ten stockouts or more are not so rare, based on the estimated probability presented in the last column of Table 15.5.

One of the advantages of a structural model like the one we estimate is that it can be used to identify the likelihood of price spikes given the current conditions. As an example, for wheat we tested the practical ability of the estimated model to predict the price spikes that occurred in the 2007-2008 food crisis. For the prediction experiment to be consistent with information available in each time period, we estimated the model to predict the prices for 2007 and 2008 considering only the sub-samples 1949-2006 and 1949-2007 respectively. The highly non-linear character of price variations, as reflected in its marked conditional skewness, implies that not only the point predictor for price in the future is conditional on the state of the system in each period, but, importantly, so is the shape of its conditional distribution.

The box plot on the right end of the graph in Figure 15.7 shows the conditional density for the price of wheat for 2007 as it would have been predicted in 2006, while Figure 15.8 shows the prediction for 2008, conditional on the information available in 2007. Far from being a complete surprise, prices as high as that of 2007 and 2008 should have been considered within the realm of possibility. The actual price of 2007 (indicated by a white star in the graphs of Figure 15.7 lays at the 85.5 percentile of the distribution of one period ahead predicted prices in 2006, and that of 2008 (the white star in the graph of Figure 15.8) at the 91.2 percentile of the predictions formed in 2007.

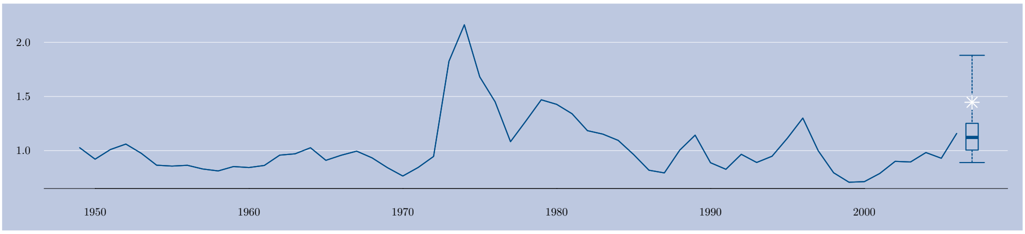
We do not claim that these would have been the best possible predictions to be formed before the actual price spike occurred. The model we present here is based on a highly simplifying assumption and only uses information on the past history of prices. Possible model refinements ought to also take into account available reliable information on the amount of stocks, if any, on production and consumption forecasts, so that the distribution of the expected net supply shock could be better calibrated. Also, more complete models should explore alternative specifications for the underlying demand function and supply response. These are all avenues to be explored in an exciting renewed research programme that, unfortunately, has been relatively idle in the past fifteen years.

Conclusions

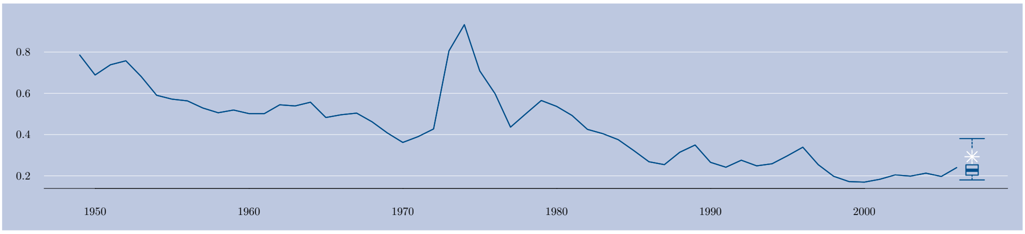
This chapter has presented the results of application of the PML estimator of the standard storage model to series of real prices of major agricultural commodities. The results have

Figure 15.7: One-step ahead prediction of wheat price (as of year 2006)

(a) De-trended series



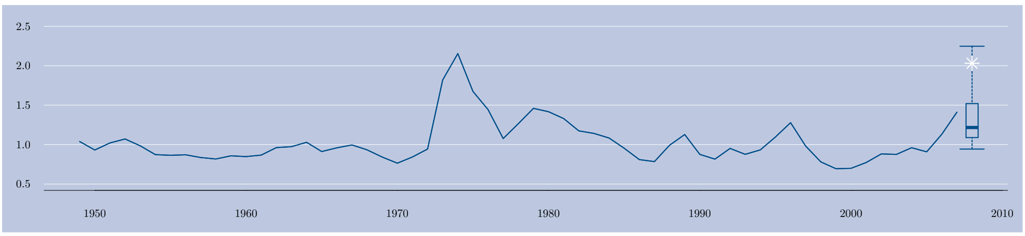
(b) Original series



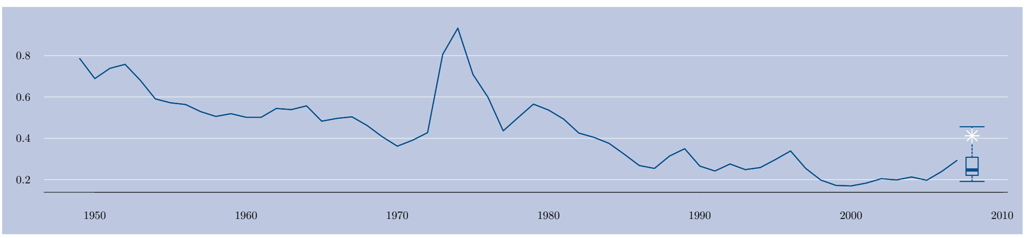
Note: The first panel shows de-trended prices. The second plot shows the original data and the conditional distribution of one period ahead price prediction, inclusive of trend.

Figure 15.8: One-step ahead prediction of wheat price (as of year 2007)

(a) De-trended series



(b) Original series



Note: The first panel shows de-trended prices. The second plot shows the original data and the conditional distribution of one period ahead price prediction, inclusive of trend.

confirmed the empirical relevance of the model. Storage arbitrage has been shown as being highly consistent with many features of the dynamics of commodity prices, including correlation and skewness. Contrary to previous findings, the evidence contained in the series of annual real price indexes does not reject even a version of the model that has been highly simplified to make it amenable to easier numerical implementation.

We have demonstrated how the model can be used to predict future prices, conditional on current information. The distributions of predicted future prices are highly skewed even when based on the assumptions of symmetric net supply shocks and linear demand, thus highlighting the possibility of price spiking above current levels.

Many areas exist for improving the estimator. Inclusion of supply response, modelling a time varying real interest rate, or exploring alternative specifications for the demand schedule - just to name a few - are all avenues to be explored in search of higher efficiency in predicting future prices. One thing is clear: a consistent treatment of the implication of intertemporal arbitrage through storage should be a fundamental ingredient of any serious analysis of commodity price behaviour.

Proof of Theorem 1

Our proof follows the structure of the proof of Theorem 1 in Deaton & Laroque (1992).

We first prove several preliminary results.

Consider $Y \equiv \{(q,z): z \in Z, q \geq \max\{F(z),0\}\}$. Let $g: Z \to [0,\infty[$ be a continuous, non-increasing function, such that $g(z) \geq F(z) \forall z \in Z$. Define $G: Y \to \mathbb{R}$ by:

$$G(q,z) \equiv \left(\frac{1-d}{1+r}\right) Eg(\omega + (1-d)x(q,z)) - k,$$

where

$$x(q,z) \equiv \begin{cases} z - F^{-1}(q), & \text{if} \quad z < z_g^* \\ z_g^* - F^{-1}(q), & \text{if} \quad z \geq z_g^* \end{cases}$$

and

$$z_g^* \equiv \inf\left\{ z \geq F^{-1}(0) : \left(\frac{1-d}{1+r}\right) Eg\Big(\omega + (1-d)(z - F^{-1}(0))\Big) - k = 0 \right\}.$$

We denote by T the operator that assigns to the function g the function Tg which satisfies the following functional equation:

$$Tg(z) = \max\{G(Tg(z),z), F(z)\}. \tag{7}$$

A SREE is a function g such that $Tg = g$.

Lemma 1. *Assume that $g: Z \to [0,\infty[$ is a continuous, non-increasing function, such that $g(z) \geq F(z), \forall z \in Z$. Then $G: Y \to \mathbb{R}$ is continuous and non-increasing in both its arguments. Furthermore, if $z < z_g^*$,*

$$G(F(z),z) = \left(\frac{1-d}{1+r}\right) Eg(\omega) - k.$$

Proof. [Proof of Lemma 1] Trivial. Note that x is continuous and g is uniformly continuous. □

Lemma 2. *Assume that g satisfies the hypotheses of Lemma 1. Then:*

1. *There exists a unique function Tg which is the solution of (7). $Tg: Z \to [0,\infty[$ is continuous, non-increasing and:*

$$\begin{aligned} Tg(z) &= F(z), && \text{for } F(z) \geq \left(\tfrac{1-d}{1+r}\right) Eg(\omega) - k \\ Tg(z) &= G(Tg(z),z), && \text{for } F(z) < \left(\tfrac{1-d}{1+r}\right) Eg(\omega) - k \end{aligned}$$

2. *Furthermore, $g_1 \geq g_2 \Rightarrow Tg_1 \geq Tg_2$.*

Proof. [Proof of Lemma 2]

1. For a given $z \in Z$, $Tg(z)$ is equal to the solution in unknown q, $q \geq \max\{F(z),0\}$, of:

$$\psi_z(q) \equiv \max\{G(q,z) - q, F(z) - q\} = 0.$$

$\psi_z(q)$ is strictly decreasing and continuous in q, and

$$\psi_z\Big(\big[\max\{F(z),0\},\infty\big[\Big) = \Big]-\infty, \psi_z\big(\max\{F(z),0\}\big)\Big].$$

To evaluate $\psi_z\big(\max\{F(z),0\}\big)$ we consider three cases:

(a) **Case 1:** For $\underline{\omega} \leq z \leq F^{-1}(0)$,

$$\psi_z\Big(\max\{F(z),0\}\Big) = \psi_z\Big(F(z)\Big) = \max\left\{\left(\frac{1-d}{1+r}\right)Eg(\omega) - k - F(z), 0\right\}.$$

If $F(z) \geq \left(\frac{1-d}{1+r}\right)Eg(\omega) - k$, then $\psi_z\Big(F(z)\Big) = 0$, and $Tg(z) = F(z)$.
If $F(z) < \left(\frac{1-d}{1+r}\right)Eg(\omega) - k$, then $\psi_z\Big(F(z)\Big) > 0$, $Tg(z)$ exists and satisfies $Tg(z) = G\Big(Tg(z), z\Big)$.

(b) **Case 2:** For $F^{-1}(0) < z < z_g^*$,

$$\psi_z\Big(\max\{F(z),0\}\Big) = \psi_z(0) = \max\{G(0,z), F(z)\} = G(0,z) > 0.$$

Then $Tg(z)$ exists and satisfies $Tg(z) = G\Big(Tg(z), z\Big)$.

(c) **Case 3:** For $z \geq z_g^*$,

$$\begin{aligned}\psi_z\Big(\max\{F(z),0\}\Big) &= \psi_z(0) = \max\{G(0,z), F(z)\} = G(0,z) \\ &= \left(\frac{1-d}{1+r}\right)Eg\Big(\omega + (1-d)(z_g^* - F^{-1}(0))\Big) - k = 0.\end{aligned}$$

Therefore $Tg(z) = 0$ and satisfies $Tg(z) = G\Big(Tg(z), z\Big)$.

This proves the existence and uniqueness of the solution $Tg(z)$. Continuity and monotonicity follow from the continuity and monotonicity of $\max\{G(q,z) - q,\ F(z) - q\}$.

2. Let g_1, g_2 be two functions that satisfy the hypotheses of Lemma 1, and such that $g_1 \geq g_2$. Then, $z_{g_1}^* \geq z_{g_2}^*$.

   (a) **Case 1:** For $z < z_{g_2}^*, G_1(q,z) \geq G_2(q,z) \quad \forall q \geq \max\{F(z),0\}$, and therefore $Tg_1(z) \geq Tg_2(z)$.

   (b) **Case 2:** For $z \geq z_{g_2}^*, Tg_2(z) = 0$, and therefore $Tg_1(z) \geq Tg_2(z)$.

□

Lemma 3. *1. If p is a SREE, and p is non-increasing in z, then $p(\underline{\omega}) = F(\underline{\omega})$.*
2. If g satisfies the assumptions of Lemma 1 and $g(\underline{\omega}) = F(\underline{\omega})$, then $Tg(\underline{\omega}) = F(\underline{\omega})$.

Proof. [Proof of Lemma 3.] Deaton & Laroque (1992, p.21). □

Proof. [Proof of Theorem 1] We can now prove theorem 1, which is composed of three parts.

1. Consider two functions g_1, g_2 satisfying the hypotheses of Lemma 1, and such that there exists a non-negative constant a with $g_2 \leq g_1 + a$.
   By Lemma 2 (2.),

$$Tg_2 \leq T(g_1 + a).$$

For $z < z_{g_1}^*$,

$$\begin{aligned}T(g_1+a)(z) &\leq \max\left\{\left(\frac{1-d}{1+r}\right)Eg_1\Big(\omega + (1-d)\Big(z - F^{-1}(Tg_1(z))\Big)\Big) - k, F(z)\right\} + \left(\frac{1-d}{1+r}\right)a \\ &= Tg_1(z) + \left(\frac{1-d}{1+r}\right)a.\end{aligned}$$

For $z_{g_1}^* \leq z < z_{g_1+a}^*$,

$$\begin{aligned}T(g_1+a)(z) &\leq \max\left\{\left(\frac{1-d}{1+r}\right)Eg_1\Big(\omega + (1-d)\Big(z_{g_1}^* - F^{-1}(0)\Big)\Big) - k, F(z)\right\} + \left(\frac{1-d}{1+r}\right)a \\ &= Tg_1(z) + \left(\frac{1-d}{1+r}\right)a.\end{aligned}$$

For $z \geq z^*_{g_1+a}$,

$$T(g_1+a)(z) = 0 \leq Tg_1(z) + \left(\frac{1-d}{1+r}\right)a.$$

Therefore, $T(g_1+a) \leq Tg_1 + \left(\frac{1-d}{1+r}\right)a$. We conclude that :

$$Tg_2 \leq Tg_1 + \left(\frac{1-d}{1+r}\right)a. \tag{8}$$

Let $\mathcal{G} \equiv \{g : Z \to [0,\infty[;\ g \text{ is continuous, non-increasing, } g \geq F,\ g(\underline{\omega}) = F(\underline{\omega})\}$.
Lemma 2 and Lemma 3 imply that $T(\mathcal{G}) \subseteq \mathcal{G}$.

$$d(g_1,g_2) \equiv \sup_{z\in Z} |g_1(z) - g_2(z)|, \quad g_1, g_2 \in \mathcal{G}$$

is a metric on $\mathcal{G}$.
For any $g_1, g_2 \in \mathcal{G}$, taking $a = d(g_1,g_2)$ in (8) we conclude that:

$$d(Tg_1, Tg_2) \leq \left(\frac{1-d}{1+r}\right) d(g_1,g_2)$$

Thus the operator T is a contraction in the complete metric space $(\mathcal{G}, d)$, and therefore it has a unique fixed point $p \in \mathcal{G}$.

2. p is strictly decreasing whenever it is strictly positive:
If not, as p is non-increasing, there is an interval where p is a positive constant. We have two cases:
   (a) **Case 1:** Suppose there exists a first interval $I \equiv [z', z'']$ where p is constant. Let $B \equiv p(z')$. $\forall\, z \in I$,

   $$B = p(z) = \left(\frac{1-d}{1+r}\right) Ep\Big(\omega + (1-d)\big(z - F^{-1}(B)\big)\Big) - k$$

   As p is non-increasing, $p\Big(\underline{\omega} + (1-d)\big(z - F^{-1}(B)\big)\Big)$ is constant $(\leq B)$, for $z \in I$. Therefore,

   $$B \leq \left(\frac{1-d}{1+r}\right) B - k, \qquad \text{a contradiction.}$$

   (b) **Case 2:** Suppose there is no first interval where p is constant. Let

   $$\mathcal{I} \equiv \{I \ :\ I \text{ is an interval where } p \text{ is constant}\}$$

   and let $\overline{p} \equiv \sup\{p(z) : z \in I \text{ and } I \in \mathcal{I}\}$.
   As there is no first interval where p is constant, $\overline{p}$ is accumulated by a sequence of values of p in I, $I \in \mathcal{I}$.
   Take any $\epsilon > 0$ and consider an interval I such that the value of p in I is $\geq \overline{p} - \epsilon$. Let $B \equiv$ value of p in I. $\forall z \in I$,

   $$B = p(z) = \left(\frac{1-d}{1+r}\right) Ep\Big(\omega + (1-d)\big(z - F^{-1}(B)\big)\Big) - k.$$

   As p is non-increasing, $p\Big(\underline{\omega} + (1-d)\big(z - F^{-1}(B)\big)\Big)$ is constant for $z \in I$ and

   $$p\Big(\underline{\omega} + (1-d)\big(z - F^{-1}(B)\big)\Big) \leq \overline{p}.$$

Therefore,

$$B \leq \left(\frac{1-d}{1+r}\right)\overline{p} - k,$$

and then,

$$B \leq \left(\frac{1-d}{1+r}\right)(B+\epsilon) - k.$$

As $\epsilon > 0$ is arbitrary, we obtain a contradiction.

3. The equilibrium level of inventories, $x(z)$, is strictly increasing for z in $\left[F^{-1}(p^*), z^*\right]$:
Let $z_1 < z_2$ in $\left[F^{-1}(p^*), z^*\right]$. As p is strictly decreasing in this interval, $p(z_1) > p(z_2)$. Therefore,

$$\left(\frac{1-d}{1+r}\right)Ep(\omega + (1-d)x(z_1)) - k > \left(\frac{1-d}{1+r}\right)Ep(\omega + (1-d)x(z_2)) - k,$$

which implies that $x(z_1) < x(z_2)$.

□

Proof of Theorem 2

The proof follows the same structure of the proof of Theorem 3 in Deaton & Laroque (1992).

Proof. The first two statements of Theorem 2 are trivial by definition. For the last part consider:

- **Claim 1:** If F is convex, then the equilibrium price function p is convex. To prove Claim 1, it suffices to prove that Tg is convex, whenever g is convex. Assume then that $g: Z \to [0, \infty[$ is a convex, continuous, non-increasing function, such that $g(z) \geq F(z)$, $\forall\, z \in Z$. Then given that F is convex and decreasing, we conclude that the associated function $G = G(q,z)$ is a convex function in the pair (q,z). Remember from the proof of Lemma 2 that for any given $z \in Z$, $Tg(z)$ is equal to the solution in unknown q, $q \geq \max\{F(z), 0\}$, of:

$$\psi_z(q) \equiv \max\{G(q,z) - q, F(z) - q\} = 0.$$

As $\psi_z(q)$ is the maximum of two convex functions, it is convex in the pair (q,z). If $,z', z'' \in Z$, and $\theta \in [0,1]$, then:

$$\psi_{z'}(q') = \psi_{z''}(q'') \;=\; 0, \quad \text{where} \quad q' = Tg(z'), \quad \text{and} \quad q'' = Tg(z'').$$

Therefore, by the convexity of $\psi_z(q)$ in (q,z), we conclude:

$$\psi_{\theta z' + (1-\theta)z''}(\theta q' + (1-\theta)q'') \;\leq\; \theta\psi_{z'}(q') \;+\; (1-\theta)\psi_{z''}(q'') = 0,$$

and given that $\psi_z(q)$ is decreasing in q, we conclude the convexity of Tg.

- **Claim 2:** If F is convex, then $p(z)$ and $x(z)$ increase when the distribution of supply shocks is modified through a mean-preserving spread.
To prove Claim 2 for a convex, continuous and non-increasing function g, let $G_1(q,z), z^*_{g,1}$ be the G function and the z^*_g point that we mention in the proof of Theorem 1 for an initial distribution of supply shocks, and let $G_2(q,z)$, $z^*_{g,2}$ be the corresponding elements for supply shocks with distribution modified through a mean preserving spread. By the convexity of g we have that $z^*_{g,1} \leq z^*_{g,2}$. If $z < z^*_{g,1}$, then $G_1(q,z) \leq G_2(q,z)$, $\forall\; q \geq \max\{F(z), 0\}$, and therefore $T_1 g(z) \leq T_2 g(z)$, where T_1, T_2 denote the corresponding contractions. If $z \geq z^*_{g,1}$, then $T_1 g(z) = 0$, and hence $T_1 g(z) \leq T_2 g(z)$, concluding

$$T_2 g(z) \;\geq\; T_1 g(z), \quad \forall\, z \in Z,$$

From this last result, and from the fact that $T_1 g$ is convex and Lemma 2 (2.), we obtain that the corresponding equilibrium price functions p_1 and p_2 satisfy:

$$p_2(z) \;\geq\; p_1(z), \quad \forall\, z \in Z.$$

□

Proof of Theorem 3

Proof. Note that $x(z)$ is continuous, non-decreasing, and bounded by $\overline{x} \equiv z^* - F^{-1}(0)$. A suitable state space for available supply is $S \equiv [\underline{\omega}, \overline{z}]$, where $\overline{z} \equiv \overline{x} + \overline{\omega} < +\infty$.

The transition probability for the available supply process is given by:

$$P[z_{t+1} \le a' \mid z_t = a] = \begin{cases} 0, & \text{if } a' < (1-d)x(a) + \underline{\omega} \\ \sum \alpha_i + (1-\alpha)\int_{\underline{\omega}}^{a'-(1-d)x(a)} m(\omega)d(\omega), & \text{if } a' \ge (1-d)x(a) + \underline{\omega} \end{cases}$$

where $\sum \alpha_i$ denotes the sum of weights of atoms in the shocks that are less than or equal to $a' - (1-d)x(a)$. The Markov operator associated to this transition probability can be written as:

$$T = \alpha T_1 + (1-\alpha) T_2,$$

where T_1 is a linear and continuous operator which takes the discrete part, and:

$$T_2 u(a) = \int_S u(z')\, m(z' - (1-d)x(a)) dz',$$

for each bounded and measurable function $u : S \to \mathbb{R}$.

In the case $\alpha < 1$, from Theorem 4.6 in Futia (1982, p. 394), T_2 is weakly compact, implying that T_2^2 is compact, and then T_2 is quasicompact. Noting that T_1 is linear and continuous, Theorem 4.10 in Futia (1982, p. 397) implies that T is quasicompact. As u continuous $\Rightarrow$ Tu continuous, by Theorem 3.3 in Futia (1982, p. 389) we conclude that T is equicontinuous. In the discrete case $\alpha = 1$, using the same results in Futia (1982, Theorem 4.6 and Theorem 3.3) we obtain that T is equicontinuous. Observing that the transition probability P satisfies the Uniqueness Criterion 2.11 in Futia with respect to the point $\underline{\omega}$, by Theorem 2.12 in Futia (1982, p. 385), we conclude that there is a unique invariant probability measure γ_* for the available supply process.

Finally, noting that the transition probability P satisfies, with respect to the point $\underline{\omega}$, what is called in Futia a Generalized Uniqueness Criterion, Theorems 3.2, 3.6, and 3.7 in Futia (1982, pp. 394 and 390) imply that, given any initial distribution γ_0 for the initial available supply, the corresponding sequence of distributions of the available supplies for the next periods $t = 1, 2, \cdots, \{\gamma_t\}_{t \in \mathbb{N}}$ converges in the total variation norm to γ_*, at a geometric rate.

Having established that the process of available supplies has a unique invariant distribution which is a global attractor, the fact that $p_t = p(z_t) = F(z_t - x(z_t))$, implies that the same holds for the Markov process of prices. □

Proof of the Remark

Proof. The facts that $F(0) = +\infty$ and $\omega_{t+1} = 0$ with positive probability, imply that if the initial available supply z_0 is positive (equivalently if the initial price p_0 is finite), then $x(z_t) > 0$, $z_t > 0$ with probability one, for all time t, and

$$p_t + k = \left(\frac{1-d}{1+r}\right) E_t p_{t+1}, \quad \forall\, t \ge 0.$$

Therefore,

$$E_0 p_t \geq \left(\frac{1+r}{1-d}\right)^t krightarrow\ +\infty, \quad \text{as } t \to +\infty.$$

This last fact together with the fact that the distribution of prices converges to a unique invariant distribution which has no atom at $+\infty$, imply that $\lim_{t\to+\infty} Var[p_t|p_0] = +\infty$.

Given any initial distribution γ_0 for z_0,

$$E[p(z_t)] \;=\; \int_S p(z_t)\gamma_t(dz_t) \;= int_S E[p(z_t)|z_0]\gamma_0(dz_0) \;\to\; +\infty, \quad \text{as } t \to +\infty.$$

Choosing $\gamma_0 = \gamma_*$, we conclude that the invariant distribution for the price process has infinite mean.

Finally, as $E[p_t] \to +\infty$, and considering that the distribution of prices converges to the unique invariant distribution which has no atom at infinite, we conclude:

$$\begin{array}{c} \forall\, \varepsilon > 0,\, \forall\, p' > 0, \quad \exists\, T \in \mathbb{N},\, \exists\, p \geq p', \quad \text{such that:} \\ t \geq T \quad \Longrightarrow \quad E[p_t] > p, \quad \text{and} \quad \text{Prob}[p_t < p] \geq 1-\varepsilon. \end{array}$$

□

Proof of the Proposition

The proof follows the structure of the proof of Proposition 1 in Deaton & Laroque (1996).

Consider the base model with discount rate r, stocks deterioration parameter d, constant marginal and average storage cost k, supply shocks ω_t, and inverse consumption demand F. By Theorem 1, there exists a unique stationary rational expectations equilibrium $p(z)$ characterized by:

$$p(z) = \max\left\{\left(\frac{1-d}{1+r}\right) Ep(\omega + (1-d)x(z)) - k,\ F(z)\right\} \tag{9}$$

where

$$x(z) = \begin{cases} z - F^{-1}(p(z)), & \text{if } z < z^* \equiv \inf\{z : p(z) = 0\} \\ z^* - F^{-1}(0), & \text{if } z \geq z^* \end{cases} \tag{10}$$

Consider the alternative model with discount rate r, stocks deterioration parameter d, constant marginal and average storage cost k, supply shocks $\widetilde{\omega}_t \equiv \sigma\omega_t + \mu$, inverse consumption demand $\widetilde{F}$ satisfying $\widetilde{F}(\sigma z + \mu) = F(z)$, and unique stationary rational expectations equilibrium $\widetilde{p}(\tilde{z})$.

Let $p_1(z) \equiv \widetilde{p}(\sigma z + \mu)$. It suffices to prove that p_1 satisfies the functional equation (9)-(10)

Proof. If $z < z_1^* \equiv \inf\{z : p_1(z) = 0\}$, then

$$Ep_1\Big(\omega + (1-d)\big(z - F^{-1}(p_1(z))\big)\Big) = E\widetilde{p}\Big(\sigma\Big[\omega + (1-d)\big(z - F^{-1}(\widetilde{p}(\sigma z + \mu))\big)\Big] + \mu\Big) =$$
$$= E\widetilde{p}\Big((\sigma\omega + \mu) + (1-d)\sigma\big(z - F^{-1}(\widetilde{p}(\tilde{z}))\big)\Big) = E\widetilde{p}\Big(\widetilde{\omega} + (1-d)\Big(\tilde{z} - \big(\widetilde{F}\big)^{-1}(\widetilde{p}(\tilde{z}))\Big)\Big),$$

where $\tilde{z} \equiv \sigma z + \mu$. Therefore,

$$\begin{aligned}
&\max\left\{\left(\frac{1-d}{1+r}\right)Ep_1\left(\omega+(1-d)\left(z-F^{-1}(p_1(z))\right)\right)-k,\, F(z)\right\}= \\
&=\max\left\{\left(\frac{1-d}{1+r}\right)E\widetilde{p}\left(\widetilde{\omega}+(1-d)\left(\tilde{z}-\left(\widetilde{F}\right)^{-1}(\widetilde{p}(\tilde{z}))\right)\right)-k,\widetilde{F}(\tilde{z})\right\}= \\
&=\widetilde{p}(\tilde{z})=\widetilde{p}(\sigma z+\mu)=p_1(z).
\end{aligned}$$

If $z \geq z_1^*$, then

$$\begin{aligned}
&Ep_1\left(\omega+(1-d)(z_1^*-F^{-1}(0))\right)= \\
&=E\widetilde{p}\left(\sigma\left[\omega+(1-d)\left(z_1^*-F^{-1}(0)\right)\right]+\mu\right)= \\
&=E\widetilde{p}\left(\widetilde{\omega}+(1-d)\left(\tilde{z}^*-\left(\widetilde{F}\right)^{-1}(0)\right)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\max\left\{\left(\frac{1-d}{1+r}\right)Ep_1\left(\omega+(1-d)(z_1^*-F^{-1}(0))\right)-k,\, F(z)\right\}= \\
&=\max\left\{\left(\frac{1-d}{1+r}\right)E\widetilde{p}\left(\widetilde{\omega}+(1-d)\left(\tilde{z}^*-\left(\widetilde{F}\right)^{-1}(0)\right)\right)-k,\widetilde{F}(\tilde{z})\right\} \\
&=\widetilde{p}(\tilde{z})=\widetilde{p}(\sigma z+\mu)=p_1(z),
\end{aligned}$$

where $\tilde{z}^* \equiv \inf\{\tilde{z} : \widetilde{p}(\tilde{z})=0\}$. □

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